II. *On the Potential of Ellipsoidal Bodies, and the Figures of Equilibrium of Rotating Liquid Masses.*

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BY an ellipsoidal body is meant, in the present paper, any homogeneous body which can be arrived at by continuous distortion of an ellipsoid. If $f_0 = 0$ is the equation of the ellipsoid from which we start, and e is a parameter, the distortion of the ellipsoid may be supposed to proceed by e increasing from the value $e = 0$ upwards, and the final figure may be taken to be

$$f_0 + ef_1 + e^2 f_2 + e^3 f_3 + \ldots = 0.$$

For very small distortions the first two terms will adequately represent the distorted figure, and as we pass to higher orders the remaining terms will enter successively.

The potential problem, to some extent interesting in itself, derives its chief importance from its application to the determination of the possible figures of equilibrium of a rotating mass of liquid. POINCARÉ,* using his ingenious method of double layers, has shown how the potential of an ellipsoidal body can be carried as far as the second-order terms when the distortion is small, but gives no indication of how it is possible to carry it further, and indeed his method is one which hardly seems susceptible of being developed further than he himself has developed it. It is clear, however, that progress with the problem of rotating liquids can only be made when a method is available for writing down the potential of an ellipsoidal body distorted as far as we please. I believe the method explained in the present paper will be found capable of giving the potential of a body distorted to any extent, although (for reasons which will be explained later) I have not in the present paper carried the calculations further than second-order terms.

The theory of figures of equilibrium of rotating masses of liquid is at present in an unsatisfactory state. It has been shown by Lord KELVIN that the Jacobian ellipsoid is stable at the point at which it coalesces with the Maclaurin spheroid, and it has been shown by POINCARÉ to remain stable up to the point at which

* "Sur la Stabilité de l'Equilibre des Figures Pyriformes affectées par une Masse Fluide en Rotation," 'Phil. Trans.,' A, vol. 198, p. 333.

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the series of Jacobian ellipsoids coalesces with the Poincaré series of pear-shaped figures. After this point the series of Jacobian ellipsoids must, in accordance with POINCARÉ'S doctrine of exchange of stabilities at a point of bifurcation, lose its stability, but the question of how it loses its stability is in a state of doubt. DARWIN believed he had proved the Poincaré series to be initially stable,* whereas LIAPOUNOFF† has maintained that this series is initially unstable. The importance of this question to theories of cosmogony is, of course, great, although perhaps liable to be overrated. A caution of POINCARÉ'S‡ may be borne in mind: "Quelle que soit l'hypothèse [stability or instability] que doive triompher un jour, je tiens à mettre toute de suite en garde contre les conséquences cosmogoniques qu'on pourrait en tirer. Les masses de la nature ne sont pas homogènes, et si on reconnaissait que les figures pyriformes sont instables, il pourrait néanmoins arriver qu'une masse hétérogène fût susceptible de prendre une forme d'équilibre analogue aux figures pyriformes, et qui serait stable. Le contraire pourrait d'ailleurs arriver également."

The present investigation was started primarily in the hope of setting this question of stability at rest. I realised that to make a new series of computations on the subject could be of little value, for whatever the result, there would have been two opinions on the one side to one on the other. Moreover, DARWIN has stated clearly that he does not think the divergence of opinion between M. LIAPOUNOFF and himself is one to be settled by new computations§: "I feel a conviction that the source of our disagreement will be found in some matter of principle." I had hoped that it might be found possible to discuss the problem by a purely algebraical method, involving neither laborious computations nor intricate physical arguments, and that if such a discussion did not give a convincing and satisfying answer to the question in hand, at least it might reveal the source of disagreement between DARWIN and LIAPOUNOFF. The result arrived at is one which, as will readily be understood from its nature, is only put forward with the utmost diffidence, but it is one from which I can find absolutely no escape. It is that underlying the whole question there is a complication, unsuspected equally by POINCARÉ, DARWIN, and (in so far as I can read his writings) LIAPOUNOFF, which renders nugatory the work of all these investigators on the stability of the pear-shaped figure. If my method is sound, it appears, as will be explained later, that it is impossible to draw any inference as to the stability of the pear from computations carried only as far as the second order of small quantities. The

* "The Stability of the Pear-shaped Figure of Equilibrium of a Rotating Mass of Liquid," 'Phil. Trans.,' 200 A (1902), p. 251; also papers in 'Phil. Trans.,' 208 A (1908), p. 1, and 'Proc. Roy. Soc., 82 A (1909), p. 188, all combined in one paper in 'Coll. Scientific Papers,' vol. 3, p. 317.

† "Sur un Problème de Tchebychef," 'Mémoires de l'Academie de St. Pétersbourg,' xvii., 3 (1905).

‡ *Loc. cit.*, p. 335.

§ 'Coll. Scientific Papers,' 3, p. 392.

materials for an answer to the question are to be found only through the third-order terms.* Fortunately the method of the present paper admits of extension to the computation of third-order terms, and so it ought to, and I hope will be quite feasible to decide as to the stability or instability of the pear, a question reserved for a subsequent paper.

The reader who is interested in the main conclusions of the paper rather than in details of theory, method, or calculations, may care to pass directly to § 35.

General Theory of Potential of Ellipsoidal Bodies.

2. We proceed to develop a method for writing down the potentials of certain homogeneous solids; in particular of ellipsoids and distorted ellipsoids. We are for the present concerned solely with potential-theory—the discussion of rotating liquids does not enter before § 19.

As will soon be evident, the problem in potential theory amounts to the following: to write the equation of the boundary of a homogeneous solid in such a form $F(x, y, z) = 0$, that the potential at the boundary is of the form $F'(x, y, z) = 0$, where $F'(x, y, z)$ is a function containing the same algebraic terms as $F(x, y, z)$, but having in general different coefficients. If this can be done, it only remains to equate $F'(x, y, z)+\frac{1}{2}\omega^2(x^2+y^2)$ to $F(x, y, z)$, and we have at once, on equating coefficients, a series of equations which will determine the possible figures of equilibrium for a liquid mass rotating with angular velocity ω.

3. Let $F(x, y, z) = 0$ be the boundary of any homogeneous solid of density ρ. Assuming it to be possible,† let V_i be a function of position satisfying $\nabla^2 V_i = -4\pi\rho$ at all points of space and coinciding with the potential of the solid at all points inside the solid, and let V_0 similarly be a function of position satisfying $\nabla^2 V_0 = 0$ at all points of space, except possibly the origin or other infinitesimal region inside the solid, and coinciding with the potential of the solid at all points outside the solid.

Then V_i must be equal to V_0 at the boundary of the solid, and we must also have $\dfrac{dV_i}{dn} = \dfrac{dV_0}{dn}$ at the boundary, where $\dfrac{d}{dn}$ denotes differentiation along the normal to the surface.

* Since writing this paper, I have been surprised to find that this conclusion is quite clearly implied in a paper which I published in 1902, "On the Equilibrium of Rotating Liquid Cylinders," 'Phil. Trans.,' A, 200, p. 67. See below, § 36.

† I have not examined in any detail the conditions that this may be possible, because the result of the paper proves that it is possible in the cases which are of importance. Similarly I have not examined in detail the difficulties which might arise at the origin or at infinity, because in the final result they do not arise. We are searching for, and ultimately find, a certain solution of the potential equations, and after the solution has been obtained it is easy to verify directly that it really is a solution, and that it involves no complications either at infinity or at the centre of the solid.

Introduce a new function W, defined by

$$W = V_i - V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

at all points of space, then we must have

$$\nabla^2 W = -4\pi\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

at all points of space, and, at the boundary, $W = 0$ and $\dfrac{dW}{dn} = 0$.

These last two conditions are equivalent to

$$\frac{dW}{dx} = \frac{dW}{dy} = \frac{dW}{dz} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

at the boundary, together with one further condition. Equations (3) require that W shall have a constant value all over the boundary; the further condition is that this constant value shall be zero.

4. Let $F(x, y, z, \lambda) = 0$ be the equation of a family of surfaces obtained by varying the parameter λ, and such that the boundary of the solid is the surface $\lambda = 0$. The surfaces of this family will divide up the solid into a series of thin shells. There will be a contribution from each shell to V_i and also to V_0. Thus W may be regarded as the sum of a number of contributions, one from each shell.

Let the thicknesses of the separate thin shells be determined by small increments in λ, say $d\lambda_1$, $d\lambda_2$, ..., starting from the boundary $\lambda = 0$. Then we may write

$$V_i = V_i(d\lambda_1) + V_i(d\lambda_2) + \ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $V_i(d\lambda_1)$ represents the contribution to V_i from the shell $d\lambda_1$, and so on. Similarly

$$V_0 = V_0(d\lambda_1) + V_0(d\lambda_2) + \ldots. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Suppose that V_i, V_0, and W are being evaluated at a point x', y', z' on the shell $d\lambda_s$ at which the value of λ is λ'. Then if $d\lambda_t$ is any shell inside the shell $d\lambda_s$, the contributions to V_i and V_0 from the shell $d\lambda_t$ will be the same; we have $V_i(d\lambda_t) = V_0(d\lambda_t)$. Hence from equations (1), (4), and (5),

$$W = V_i - V_0 = \{V_i(d\lambda_1) - V_0(d\lambda_1)\} + \{V_i(d\lambda_2) - V_0(d\lambda_2)\} + \ldots + \{V_i(d\lambda_s) - V_0(d\lambda_s)\}, \quad (6)$$

or expressed as an integral,

$$W = \int_0^{\lambda'} \Phi(x', y', z', \lambda)\, d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

5. This form for W satisfies automatically the last of the conditions of § 3, namely, that W shall vanish at the boundary. We proceed to determine Φ so as to satisfy the remaining conditions which are expressed by equations (2) and (3).

Let $\frac{d}{dx}$ be used to denote geometrical differentiation with respect to x—*i.e.*, differentiation in space keeping y and z constant—and let $\frac{\partial}{\partial x}$ be used to denote algebraic differentiation, *i.e.*, partial differentiation with respect to x keeping y, z, and λ constant. Then

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{d\lambda}{dx}\frac{\partial}{\partial \lambda}, \text{ \&c.,} \qquad (8)$$

and we have

$$\begin{aligned}\frac{dW}{dx'} &= \frac{\partial W}{\partial x'} + \frac{d\lambda'}{dx'}\frac{\partial W}{|\partial \lambda'|} \\ &= \frac{\partial W}{\partial x'} + \frac{d\lambda'}{dx'}\Phi(x', y', z', \lambda'). \qquad (9)\end{aligned}$$

If the point x', y', z' is on the boundary we have $\lambda' = 0$; so that from equation (7), $\frac{\partial W}{\partial x'} = 0$, and it appears that equations (3) will all be satisfied if

$$\Phi(x', y', z', \lambda') = 0, \qquad (10)$$

at all points on the boundary $\lambda' = 0$.

We notice from a comparison of equations (6) and (7) that

$$\Phi(x', y', z', \lambda')\, d\lambda_s = V_i(d\lambda_s) - V_0(d\lambda_s),$$

and the right hand of course vanishes when x', y', z' is on the shell $d\lambda_s$. Thus we must have, at all points,

$$\Phi(x', y', z', \lambda') = 0 \qquad (11)$$

identically, provided that λ' has the value appropriate to the point x', y', z'. This condition of course imposes more restriction on the value of Φ than does equation (10). Equation (10) was adequate to ensure that the boundary condition (3) should be satisfied, but the remark just made shows that for (3) and (2) both to be satisfied—*i.e.*, for W to give the true value of $V_i - V_0$, equation (11) must necessarily be satisfied. We shall now assume that equation (11) is satisfied, and proceed to satisfy the remaining condition expressed by equation (2), namely,

$$\nabla^2 W = -4\pi\rho. \qquad (12)$$

In virtue of equation (11), equation (9) reduces to

$$\frac{dW}{dx'} = \frac{\partial W}{\partial x'},$$

whence on further differentiation (*cf.* equation (8)),

$$\frac{d^2W}{dx'^2} = \frac{\partial^2 W}{\partial x'^2} + \frac{d\lambda'}{dx'}\frac{\partial^2 W}{\partial\lambda'\partial x'} = \frac{\partial^2 W}{\partial x'^2} + \frac{d\lambda'}{dx'}\frac{\partial}{\partial x'}\Phi(x', y', z', \lambda'),$$

so that equation (12) becomes

$$-4\pi\rho = \frac{d^2W}{dx'^2} + \frac{d^2W}{dy'^2} + \frac{d^2W}{dz'^2}$$

$$= \int_0^{\lambda'}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)\Phi(x', y', z', \lambda)\,d\lambda + \Sigma\frac{d\lambda'}{dx'}\frac{\partial}{\partial x'}\Phi(x', y', z', \lambda'). \quad . \quad . \quad (13)$$

From equations (8) and (11),

$$\frac{\partial}{\partial x'}\Phi(x', y', z', \lambda') + \frac{d\lambda'}{dx'}\frac{\partial}{\partial\lambda'}\Phi(x', y', z', \lambda') = 0,$$

so that equation (13) may be written in either of the equivalent forms:

$$-4\pi\rho = \int_0^{\lambda'}\nabla^2\Phi\,d\lambda - \left\{\left(\frac{d\lambda'}{dx'}\right)^2 + \left(\frac{d\lambda'}{dy'}\right)^2 + \left(\frac{d\lambda'}{dz'}\right)^2\right\}\frac{\partial\Phi'}{\partial\lambda'}, \quad . \quad . \quad . \quad . \quad (14)$$

$$-4\pi\rho = \int_0^{\lambda'}\nabla^2\Phi\,d\lambda - \frac{\left(\frac{\partial\Phi'}{\partial x'}\right)^2 + \left(\frac{\partial\Phi'}{\partial y'}\right)^2 + \left(\frac{\partial\Phi'}{\partial z'}\right)^2}{\frac{\partial\Phi'}{\partial\lambda'}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

in which ∇^2 stands for $\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}$, Φ for $\Phi(x', y', z', \lambda)$ and Φ' for $\Phi(x', y', z', \lambda')$.

Thus if Φ satisfies either equation (14) or (15), and also equation (11), then W, as given by equation (7), satisfies all the conditions which have been seen to be necessary, and will therefore give the true value of $V_i - V_0$.*

* Suppose there are, if possible, two solutions to the same problem, say $\Phi = \Phi_1$ and $\Phi = \Phi_2$. Since W is determined when the problem is fixed, we must have

$$W = \int_0^{\lambda'}\Phi_1\,d\lambda = \int_0^{\lambda'}\Phi_2\,d\lambda,$$

so that

$$\int_0^{\lambda'}(\Phi_1 - \Phi_2)\,d\lambda = 0, \quad \text{or} \quad \Phi_1 = \Phi_2 + \frac{\partial\chi}{\partial\lambda}$$

where

$$\chi(x', y', z', \lambda') - \chi(x', y', z', 0) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

for all values of x', y', z'. Thus, if χ is such as to satisfy (i) we may add a term $\frac{\partial\chi}{\partial\lambda}$ to Φ and still obtain a solution of the same problem. A special case in which (i) is satisfied is when

$$\chi(x', y', z', \lambda') = f(x', y', z', \lambda')\{\psi(\lambda') - \psi(0)\},$$

where f is any function which vanishes identically for the value of λ' appropriate to the point x', y', z',

6. As a matter of convenience, involving neither loss nor gain of generality, we shall write

$$\Phi(x, y, z, \lambda) = \psi(\lambda) f(x, y, z, \lambda), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

in which $\psi(\lambda)$ is any function of λ, and $f(x, y, z, \lambda)$ a quite general function of x, y, z, and λ. Then, from equation (11), we must have

$$f(x, y, z, \lambda) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

identically at all points, when λ has the value appropriate to the point x, y, z. We accordingly have

$$\frac{\partial \Phi}{\partial \lambda} = \psi(\lambda) \frac{\partial f}{\partial \lambda}, \qquad \frac{\partial \Phi}{\partial x} = \psi(\lambda) \frac{\partial f}{\partial x}, \text{ \&c.,}$$

so that equations (14) and (15) reduce to

$$-4\pi\rho = \int_0^{\lambda'} \psi(\lambda) \nabla^2 f \, d\lambda - \psi(\lambda') \left\{ \left(\frac{d\lambda'}{dx'}\right)^2 + \left(\frac{d\lambda'}{dy'}\right)^2 + \left(\frac{d\lambda'}{dz'}\right)^2 \right\} \frac{\partial f'}{\partial \lambda'} \quad . \quad . \quad . \quad (18)$$

$$-4\pi\rho = \int_0^{\lambda'} \psi(\lambda) \nabla^2 f \, d\lambda - \psi(\lambda') \frac{\left(\frac{\partial f'}{\partial x'}\right)^2 + \left(\frac{\partial f'}{\partial y'}\right)^2 + \left(\frac{\partial f'}{\partial z'}\right)^2}{\frac{\partial f'}{\partial \lambda'}} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Moreover the family of surfaces (λ = cons.) may now be supposed to be determined by equation (17), and the boundary will be given by

$$f(x, y, z, 0) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Thus, to sum up, if f and ψ are such that either equation (18) or (19) is satisfied, then the potential of the homogeneous solid of density ρ whose boundary is determined by equation (20), will be given by

$$V_i - V_0 = W = \int_0^{\lambda'} \psi(\lambda) f(x', y', z', \lambda) \, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

the value of W being evaluated at the point x', y', z', and λ' being determined from the equation $f(x', y', z', \lambda') = 0$.

7. The boundary $\lambda = 0$ is of course fixed by the solid whose potential is required, but we are left with a certain amount of choice as to the disposition of the surfaces

and ψ is any function of λ whatever. Replacing $\psi(\lambda') - \psi(0)$ by $\int_0^{\lambda'} u(\lambda) \, d\lambda$, we find that if Φ is a solution then

$$\Phi + \frac{\partial}{\partial \lambda'} \left\{ f \int_0^{\lambda'} u(\lambda) \, d\lambda \right\}$$

will also be a solution of the same problem.

$\lambda =$ cons. We shall now limit this amount of freedom by assuming that the region at infinity is made to coincide with the surface $\lambda = +\infty$.

Consider a new function V'_i defined at any point of space x', y', z', by

$$V'_i = \int_0^\infty \psi(\lambda) f(x', y', z', \lambda)\, d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

then

$$\begin{aligned} \nabla^2 V'_i &= \int_0^\infty \psi(\lambda)\, \nabla^2 f\, d\lambda \\ &= -4\pi\rho + \left[\psi(\lambda')\left(\frac{d\lambda'}{dn'}\right)^2 \frac{\partial f'}{\partial \lambda'}\right]_{\lambda'=\infty} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23) \end{aligned}$$

by equation (18). Hence, if (as will be the case in all our applications) the value of the limit when $\lambda' = \infty$ of the term in square brackets is zero, we shall have

$$\nabla^2 V'_i = -4\pi\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

At infinity V_0 must vanish, so that at infinity, by equation (21),

$$V_i = W = \int_0^\infty \psi(\lambda) f(x', y', z', \lambda)\, d\lambda = V'_i.$$

Thus $V'_i - V_i$ vanishes at infinity, and satisfies $\nabla^2 (V'_i - V_i) = 0$ at all points of space; whence (except for a possible singularity at the origin, which will be found not to cause trouble) we must have $V'_i = V_i$, so that V'_i is the internal potential. Knowing V_i and W we find V_0 immediately by equation (1), and have

$$V_i = \int_0^\infty \psi(\lambda) f(x', y', z', \lambda)\, d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$V_0 = \int_{\lambda'}^\infty \psi(\lambda) f(x', y', z', \lambda)\, d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

To recapitulate, the condition that these equations shall give the true values of the potential are

(i) that V_i shall be finite at the origin,

(ii) that $\psi(\lambda)\left(\frac{d\lambda}{dn}\right)^2 \frac{\partial f}{\partial \lambda}$ shall vanish at infinity.

If these conditions are satisfied, as they will be without trouble in all our applications, then equations (25) and (26) will give the potentials.

8. If $f(x, y, z, 0)$ is the equation of the boundary, the potential at the boundary will be

$$\int_0^\infty \psi(\lambda) f(x, y, z, \lambda)\, d\lambda,$$

and therefore will contain exactly the same terms in x, y, z as the equation of the

boundary, but with different coefficients. The method is therefore exactly suited for the determination of figures of equilibrium of rotating fluid (*cf.* § 2).

As a method of determining potentials, the procedure is indirect in the sense that we cannot pass by any direct series of processes from the equation of the boundary of the solid, as expressed by equation (20), to the general function $f(x, y, z, \lambda)$. We must first search for solutions of equations (18) or (19), and then examine what problem is solved.

An obvious case to examine first is that in which f is an algebraic function of the second degree. In this case $\nabla^2 f$ is a function of λ only, so that the equation for f can be satisfied if the last term in equation (18) or (19) is a function of λ only.

Examples of General Theory.

I. *A Sphere.*

9. A quite trivial example may perhaps be taken first, namely that of the sphere $x^2+y^2+z^2 = a^2$. It is seen without trouble that any way of forming the function $f(x, y, z, \lambda)$ will lead to a solution, provided that this function involves x, y, z only through $x^2+y^2+z^2$,—*i.e.*, provided the surfaces are taken to be concentric spheres. For instance, we may take

$$f(x, y, z, \lambda) = x^2+y^2+z^2-(\lambda-a)^2,$$

then equation (18) reduces to

$$-4\pi\rho = \int_0^{\lambda'} 6\psi(\lambda)\, d\lambda + 2(\lambda'-a)\,\psi(\lambda'),$$

of which the solution is found to be $\psi(\lambda) = \dfrac{2\pi\rho a^3}{(\lambda-a)^4}$.

The potentials are now given by

$$\mathrm{V}_i = \int_0^\infty \frac{2\pi\rho a^3}{(\lambda-a)^4}\{x^2+y^2+z^2-(\lambda-a)^2\}\, d\lambda = \frac{\mathrm{M}}{a} - \frac{\mathrm{M}}{2a^3}(r^2-a^2)$$

$$\mathrm{V}_0 = \int_\lambda^\infty \frac{2\pi\rho a^3}{(\lambda-a)^4}\{x^2+y^2+z^2-(\lambda-a)^2\}\, d\lambda = \frac{\mathrm{M}}{r},$$

where M is written for $\frac{4}{3}\pi\rho a^3$, and $r^2 = x^2+y^2+z^2 = (\lambda-a)^2$.

II. *An Ellipsoid.*

10. It is readily seen that a solution of equation (19) can be obtained by taking

$$f(x, y, z, \lambda) = \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The equation reduces to

$$-4\pi\rho = \int_0^{\lambda'} \psi(\lambda) \left\{\frac{2}{a^2+\lambda} + \frac{2}{b^2+\lambda} + \frac{2}{c^2+\lambda}\right\} d\lambda + 4\psi(\lambda'), \quad . \quad . \quad . \quad . \quad (28)$$

of which the solution is readily found to be

$$\psi(\lambda) = -\frac{\pi\rho abc}{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{1/2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

These values are found to satisfy the conditions of § 7, so that the potentials are given by

$$V_i = -\pi\rho abc \int_0^\infty \frac{1}{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{1/2}} \left(\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1\right) d\lambda,$$

$$V_0 = -\pi\rho abc \int_\lambda^\infty \frac{1}{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{1/2}} \left(\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1\right) d\lambda.$$

III. *A Distorted Ellipsoid.*

11. For an ellipsoid distorted in any way, and without any limitation (at present) as to the distortion being small, assume

$$\Phi(x, y, z, \lambda) = \psi(\lambda)(f+\phi)$$

where $\psi(\lambda)$ and f have the same meanings as before, being given by equations (27) and (29), and ϕ is any general function of x, y, z, and λ. The boundary of the distorted ellipsoid is of course

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 + \phi_{\lambda=0} = 0.$$

Equation (19) becomes

$$-4\pi\rho = \int_0^\lambda \psi(\lambda)\left\{\frac{2}{A} + \frac{2}{B} + \frac{2}{C} + \nabla^2\phi\right\} d\lambda + \psi(\lambda) \frac{\left(\frac{2x}{A} + \frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{2y}{B} + \frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{2z}{C} + \frac{\partial\phi}{\partial z}\right)^2}{\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} - \frac{\partial\phi}{\partial\lambda}},$$

in which x', y', z', λ' have been replaced by x, y, z, λ now that there is no danger of confusion, and A, B, C are written for $a^2 + \lambda$, $b^2 + \lambda$, $c^2 + \lambda$. If we further put

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} \equiv -\frac{\partial f}{\partial\lambda} = \varpi^2,$$

the equation becomes

$$-4\pi\rho\left(\varpi^2 - \frac{\partial\phi}{\partial\lambda}\right) = \left(\varpi^2 - \frac{\partial\phi}{\partial\lambda}\right)\int_0^\lambda \psi(\lambda)\left\{\frac{2}{\mathrm{A}} + \frac{2}{\mathrm{B}} + \frac{2}{\mathrm{C}} + \nabla^2\phi\right\} d\lambda$$

$$+ \psi(\lambda)\left\{4\varpi^2 + 4\Sigma\frac{x}{\mathrm{A}}\frac{\partial\phi}{\partial x} + \Sigma\left(\frac{\partial\phi}{\partial x}\right)^2\right\}.$$

But from equation (28)

$$-4\pi\rho\left(\varpi^2 - \frac{\partial\phi}{\partial\lambda}\right) = \left(\varpi^2 - \frac{\partial\phi}{\partial\lambda}\right)\int_0^\lambda \psi(\lambda)\left\{\frac{2}{\mathrm{A}} + \frac{2}{\mathrm{B}} + \frac{2}{\mathrm{C}}\right\} d\lambda + 4\left(\varpi^2 - \frac{\partial\phi}{\partial\lambda}\right)\psi(\lambda),$$

so that on subtraction we find as the equation to be satisfied by ϕ,

$$\left(\varpi^2 - \frac{\partial\phi}{\partial\lambda}\right)\int_0^\lambda \psi(\lambda)\nabla^2\phi\, d\lambda + \psi(\lambda)\left\{4\Sigma\frac{x}{\mathrm{A}}\frac{\partial\phi}{\partial x} + 4\frac{\partial\phi}{\partial\lambda} + \Sigma\left(\frac{\partial\phi}{\partial x}\right)^2\right\} = 0. \quad . \quad . \quad (30)$$

The equation is too complicated to be attacked directly, but can be effectively broken up by assuming a solution

$$\phi = u + fv,$$

in which u and v are general functions of x, y, z, and λ, while f is given by equation (27). On substituting this value for ϕ, equation (30) reduces, after considerable simplification, to

$$\left(\varpi^2 - \frac{\partial\phi}{\partial\lambda}\right)\int_0^\lambda \psi(\lambda)\nabla^2(u + fv)\, d\lambda$$

$$+\psi(\lambda)\left\{4(1+v)\left[\left(\Sigma\frac{x}{\mathrm{A}}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial\lambda}\right) + f\left(\Sigma\frac{x}{\mathrm{A}}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial\lambda}\right)\right]\right.$$

$$\left. + 4\left(\varpi^2 - \frac{\partial\phi}{\partial\lambda}\right)v + \Sigma\left(\frac{\partial u}{\partial x} + f\frac{\partial v}{\partial x}\right)^2\right\} = 0,$$

and this will be satisfied if we satisfy separately the equations

$$\int_0^\lambda \psi(\lambda)\nabla^2(u+fv)\, d\lambda + 4\psi(\lambda)v = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$4(1+v)\left[\left(\Sigma\frac{x}{\mathrm{A}}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial\lambda}\right) + f\left(\Sigma\frac{x}{\mathrm{A}}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial\lambda}\right)\right] + \Sigma\left(\frac{\partial u}{\partial x} + f\frac{\partial v}{\partial x}\right)^2 = 0. \quad . \quad . \quad (32)$$

On substituting for f and ψ, and writing $\mathrm{ABC} = \Delta^2$, equation (31) becomes

$$\frac{4v}{\Delta} = -\int_0^\lambda \left\{\nabla^2 u + f\nabla^2 v + \Sigma\frac{4x}{\mathrm{A}}\frac{\partial v}{\partial x} + 2v\left(\frac{1}{\mathrm{A}} + \frac{1}{\mathrm{B}} + \frac{1}{\mathrm{C}}\right)\right\}\frac{d\lambda}{\Delta}$$

in which

$$-\int_0^\lambda 2v\left(\frac{1}{\mathrm{A}} + \frac{1}{\mathrm{B}} + \frac{1}{\mathrm{C}}\right)\frac{d\lambda}{\Delta} = \int_0^\lambda 4v\frac{\partial}{\partial\lambda}\left(\frac{1}{\Delta}\right) d\lambda$$

$$= \frac{4v}{\Delta} - \left(\frac{4v}{\Delta}\right)_{\lambda=0} - \int_0^\lambda \frac{4}{\Delta}\frac{\partial v}{\partial\lambda}\, d\lambda.$$

Thus equation (31) is equivalent to

$$\int_0^\lambda \left\{\nabla^2 u + f\nabla^2 v + 4\left(\Sigma\frac{x}{A}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial \lambda}\right)\right\}\frac{d\lambda}{\Delta} = \left|\frac{4v}{\Delta}\right|_{\lambda=0}. \qquad (33)$$

This must be satisfied for all values of λ, so that we must have (as is clear on putting $\lambda = 0$ in the equation) $v = 0$ when $\lambda = 0$.

It will be remembered that the boundary of the distorted figure is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 + \phi_{\lambda=0} = 0,$$

and it is now clear that $\phi_{\lambda=0}$ reduces to $u_{\lambda=0}$. Thus the generality of the boundary must be involved in the generality of u, and provided u is kept general, we shall obtain a general solution of the problem, even if we take the simplest possible value for v. The most general way of satisfying equation (33) is to take

$$\nabla^2 u + f\nabla^2 v + 4\left(\Sigma\frac{x}{A}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial \lambda}\right) \equiv \Delta\frac{\partial \chi}{\partial \lambda} \qquad (34)$$

where χ may be any function of x, y, z, and λ which vanishes for λ or 0, but the simplest way of satisfying the equation is to take

$$\nabla^2 u + f\nabla^2 v + 4\left(\Sigma\frac{x}{A}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial \lambda}\right) \equiv 0. \qquad (35)$$

In each of these equations the sign of identity ($\equiv$) is used in place of the sign of equality, because in the integrand of equation (33) the value of λ is not the same as the value of λ in the upper limit of the integral, which is determined by the values of x, y, z.

12. To shorten the algebra we may change to a new set of variables λ, ξ, η, ζ connected with the old variables λ, x, y, z by the relations

$$\lambda = \lambda, \qquad \xi = \frac{x}{A}, \qquad \eta = \frac{y}{B}, \qquad \zeta = \frac{z}{C}.$$

Differentiation with respect to the new variable λ is given by

$$\frac{\partial}{\partial\lambda_{\text{new}}} = \frac{\partial}{\partial\lambda_{\text{old}}} + \Sigma\frac{\partial x}{\partial\lambda}\frac{\partial}{\partial x}$$

where, since $x = (a^2+\lambda)\,\xi$, we have $\dfrac{\partial x}{\partial\lambda} = \xi = \dfrac{x}{A}$, and so

$$\frac{\partial}{\partial\lambda_{\text{new}}} = \frac{\partial}{\partial\lambda_{\text{old}}} + \Sigma\frac{x}{A}\frac{\partial}{\partial x}. \qquad (36)$$

The old symbol ∇^2 has been taken to mean $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)_{\lambda = \text{cons.}}$, which in the new variables becomes

$$\frac{1}{A^2}\frac{\partial^2}{\partial \xi^2}+\frac{1}{B^2}\frac{\partial^2}{\partial \eta^2}+\frac{1}{C^2}\frac{\partial^2}{\partial \zeta^2},$$

and this will be denoted by $\nabla^2_{\xi\eta\zeta}$.

Equations (32) and (35) in the new co-ordinates, reduce to

$$4(1+v)\left(\frac{\partial u}{\partial \lambda}+f\frac{\partial v}{\partial \lambda}\right)+\Sigma\frac{1}{A^2}\left(\frac{\partial u}{\partial \xi}+f\cdot\frac{\partial v}{\partial \xi}\right)^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$\nabla^2_{\xi\eta\zeta}u+f\nabla^2_{\xi\eta\zeta}v+4\frac{\partial v}{\partial \lambda} \equiv 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

We are assuming that $f+\phi = 0$ when λ has the value appropriate to the values of ξ, η, ζ, so that in the first equation f may be replaced by $-\phi$, but in the second equation this may not be done.

13. It is convenient, for the purposes of the present paper, to suppose the distortion to start from the undistorted ellipsoid, and to proceed in powers of a parameter e. Thus we assume

$$u = eu_1+e^2u_2+e^3u_3+\ \dots\ ,$$

$$v = ev_1+e^2v_2+e^3v_3+\ \dots\ ,$$

and in the equation (37) since $f+(u+fv) = 0$, it is clear that, when e is small, f will be a small quantity of the smallness of e. As far as e, equation (37) reduces to $\frac{\partial u_1}{\partial \lambda} = 0$, giving u_1. Equation (38) then gives v_1; equation (37) taken as far as e^2 will then give u_2, and (38) will give v_2; (37) taken as far as e^3 will give u_3, (38) will give v_3 and so on.

As far as e only, equation (37) reduces to $\frac{\partial u_1}{\partial \lambda} = 0$, of which the solution is $u_1 = \chi(\xi, \eta, \zeta)$ where χ is the most general function of ξ, η, and ζ. At the boundary ϕ reduces to $(u_1)_{\lambda=0}$ or to $\chi\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$, so that the generality of the function χ enables us to deal with the most general small displacement possible.

At present we shall consider only solutions for which χ is algebraic and of degree not greater than 3 in ξ, η, ζ. For these solutions equation (38) shows that v_1 will be of degree not greater than 1 in ξ, η, ζ, so that $\nabla^2 v_1 = 0$, and the equation reduces to

$$4\frac{\partial v_1}{\partial \lambda} = -\nabla^2_{\xi\eta\zeta}u_1 = -\left(\frac{1}{A^2}\frac{\partial^2}{\partial \xi^2}+\frac{1}{B^2}\frac{\partial^2}{\partial \eta^2}+\frac{1}{C^2}\frac{\partial^2}{\partial \zeta^2}\right)u_1$$

or

$$4v_1 = \left\{\left(\frac{1}{A} - \frac{1}{a^2}\right)\frac{\partial^2}{\partial \xi^2} + \left(\frac{1}{B} - \frac{1}{b^2}\right)\frac{\partial^2}{\partial \eta^2} + \left(\frac{1}{C} - \frac{1}{c^2}\right)\frac{\partial^2}{\partial \zeta^2}\right\} u_1, \quad . \quad . \quad . \quad (39)$$

since v must vanish when $\lambda = 0$.

Remembering that $\frac{\partial u_1}{\partial \lambda} = 0$, and that $f = -eu_1 + \ldots$, the terms in e^2 in (37) now give

$$4\frac{\partial u_2}{\partial \lambda} = 4u_1\frac{\partial v_1}{\partial \lambda} - \Sigma \frac{1}{A^2}\left(\frac{\partial u_1}{\partial \xi}\right)^2$$

$$= -\Sigma \frac{1}{A^2}\left\{u_1 \frac{\partial^2 u_1}{\partial \xi} + \left(\frac{\partial u_1}{\partial \xi}\right)^2\right\}$$

$$= -\tfrac{1}{2}\Sigma \frac{1}{A^2}\frac{\partial^2}{\partial \xi^2}(u_1^2),$$

giving on integration

$$4u_2 = -\tfrac{1}{2}\int_0^\lambda \left(\frac{1}{A^2}\frac{\partial^2}{\partial \xi^2} + \frac{1}{B^2}\frac{\partial^2}{\partial \eta^2} + \frac{1}{C^2}\frac{\partial^2}{\partial \zeta^2}\right) u_1^2 \, d\lambda + \omega(\xi, \eta, \zeta), \quad . \quad . \quad . \quad (40)$$

in which ω is again the most general function possible of ξ, η, ζ (enabling us to carry on the distortion to the second order in any way we please), and the lower limit of the integral is taken to be zero simply as a matter of convenience.

The addition of a perfectly general function ω would be equivalent to the superposition of a perfectly general distortion (proportional to e^2) on to the distortion already under consideration. The real object of the present analysis is to be found in its ultimate application to the problem of the rotating fluid, and to solve this problem, it will be found that ω need contain no terms of degree higher than 4 in ξ, η, ζ, this being also the degree of the other terms in u_2. Hence in what follows it will be supposed that u_2 contains no terms of degree higher than 4 in ξ, η, ζ.

A value of v_2 is obtainable from equation (38), but there are, as has been seen, many possible forms for v_2, and the most convenient is, in point of fact, obtained by going back to equation (34), which in x, y, z co-ordinates is

$$\nabla^2 u_2 + f\nabla^2 v_2 + 4\left(\Sigma \frac{x}{A}\frac{\partial v_2}{\partial x} + \frac{\partial v_2}{\partial \lambda}\right) \equiv \Delta \frac{\partial \chi}{\partial \lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where χ may be any function of x, y, z, and λ which vanishes (to the power of e we are now concerned with) both for λ and 0.

Let two new functions w and w' be introduced, defined by

$$4\left(\frac{\partial w}{\partial \lambda} + \Sigma \frac{x}{A}\frac{\partial w}{\partial x}\right) = -\nabla^2 u_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

$$4\left(\frac{\partial w'}{\partial \lambda} + \Sigma \frac{x}{A}\frac{\partial w'}{\partial x}\right) = -\tfrac{1}{2}\nabla^2 w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Clearly since u_2 is of degree 4 in x, y, z, w will be of degree 2, so that w' will be a function of λ only. The term $\Sigma \frac{x}{A}\frac{\partial w'}{\partial x}$ in equation (43) is therefore zero in the present instance, but is inserted to maintain symmetry. We now have

$$4\left(\frac{\partial}{\partial\lambda}+\Sigma\frac{x}{A}\frac{\partial}{\partial x}\right)(w+fw')$$
$$= 4w'\left(\frac{\partial}{\partial\lambda}+\Sigma\frac{x}{A}\frac{\partial}{\partial x}\right)f + 4\left(\frac{\partial}{\partial\lambda}+\Sigma\frac{x}{A}\frac{\partial}{\partial x}\right)w+4f\left(\frac{\partial}{\partial\lambda}+\Sigma\frac{x}{A}\frac{\partial}{\partial x}\right)w'$$
$$= 4w'\varpi^2-\nabla^2 u_2-\tfrac{1}{2}f\nabla^2 w,$$

so that after simplification,

$$\nabla^2 u_2+f\nabla^2(w+fw')+4\left(\frac{\partial}{\partial\lambda}+\Sigma\frac{x}{A}\frac{\partial}{\partial x}\right)(\omega+fw'),$$
$$=\tfrac{1}{2}f\nabla^2 w+w'f\nabla^2 f+4w'\varpi^2,$$
$$=-4\Delta\frac{\partial}{\partial\lambda}\left(\frac{fw'}{\Delta}\right).$$

Since f vanishes for the appropriate value of λ, $\frac{fw'}{\Delta}$ will vanish for both λ and 0 provided w' is made to vanish when $\lambda = 0$. Thus $\frac{fw'}{\Delta}$ will satisfy the condition to be satisfied by χ in equation (41), and a solution of this equation will be

$$v_2 = w+fw'. \qquad (44)$$

Since v_2 must vanish when $\lambda = 0$ (§ 11) it appears that both w and w' must vanish separately when $\lambda = 0$. On transforming (41) and (42) to ξ, η, ζ co-ordinates (*cf.* equation (36)) and integrating, we obtain as the values of w and w' which vanish when $\lambda = 0$,

$$w = -\tfrac{1}{4}\int_0^\lambda \nabla^2_{\xi\eta\zeta}u_2\,d\lambda\,; \qquad w' = -\tfrac{1}{8}\int_0^\lambda \nabla^2_{\xi\eta\zeta}w\,d\lambda. \qquad (45)$$

14. Let us introduce a differential operator D, defined by

$$D = \left(\frac{1}{a^2}-\frac{1}{A}\right)\frac{\partial^2}{\partial\xi^2}+\left(\frac{1}{b^2}-\frac{1}{B}\right)\frac{\partial^2}{\partial\eta^2}+\left(\frac{1}{c^2}-\frac{1}{C}\right)\frac{\partial^2}{\partial\zeta^2}, \qquad (46)$$

noticing that, as a function of λ, D is purely a multiplier. We have

$$\frac{\partial D}{\partial\lambda} = \Sigma\frac{1}{A^2}\frac{\partial^2}{\partial\xi^2} = \nabla^2_{\xi\eta\zeta},$$

and when $\lambda = 0$, $D = 0$.

The value of v_1 already obtained (equation (39)) is

$$v_1 = -\tfrac{1}{4}\mathrm{D}u_1, \qquad (47)$$

and the value of u_2 (equation (40)) is

$$u_2 = \tfrac{1}{4}\omega(\xi, \eta, \zeta) - \tfrac{1}{8}\int_0^\lambda \frac{\partial}{\partial\lambda}\,\mathrm{D}u_1^2\,d\lambda$$

$$= \tfrac{1}{4}\omega(\xi, \eta, \zeta) - \tfrac{1}{8}\mathrm{D}u_1^2. \qquad (48)$$

Hence from equations (45)

$$w = -\tfrac{1}{4}\int_0^\lambda \nabla^2_{\xi\eta\zeta}\, u_2\, d\lambda,$$

$$= -\tfrac{1}{4}\int_0^\lambda \frac{\partial\mathrm{D}}{\partial\lambda}\left(\tfrac{1}{4}\omega - \tfrac{1}{8}\mathrm{D}u_1^2\right) d\lambda,$$

$$= -\tfrac{1}{16}\mathrm{D}\omega + \tfrac{1}{64}\mathrm{D}^2 u_1^2. \qquad (49)$$

$$w' = -\tfrac{1}{8}\int_0^\lambda \nabla^2_{\xi\eta\zeta}\, w\, d\lambda$$

$$= -\tfrac{1}{8}\int_0^\lambda \frac{\partial\mathrm{D}}{\partial\lambda}\left(-\tfrac{1}{16}\mathrm{D}\omega + \tfrac{1}{64}\mathrm{D}^2 u_1^2\right) d\lambda$$

$$= \tfrac{1}{256}\mathrm{D}^2\omega - \tfrac{1}{1536}\mathrm{D}^3 u_1^2.$$

15. This completes the solution as far as the second order of small quantities. We shall not attempt to evaluate u_3 and v_3, as the problems discussed in the present paper require a solution as far as e^2 only.

As far as e^2, the value of ϕ has been seen to be given by

$$\phi = u + fv = e(u_1 + fv_1) + e^2(u_2 + fw + f^2 w') \qquad (50)$$

and the potentials can now be found directly from the formula (§ 11)

$$\mathrm{W} = \int_0^\lambda \psi(\lambda)(f + \phi)\, d\lambda.$$

As in § 7, examine a function V'_i defined by

$$\mathrm{V}'_i = \int_0^\infty \psi(\lambda)(f + \phi)\, d\lambda,$$

then

$$\nabla^2\mathrm{V}'_i = \int_0^\infty \psi(\lambda)\{\nabla^2 f + \nabla^2(u + fv)\}\, d\lambda. \qquad (51)$$

Now the value of $\int_0^\infty \psi(\lambda)\nabla^2 f\, d\lambda$ is by § 10, equal to $-4\pi\rho$, while from equation (31)

we have

$$\int_0^\infty \psi(\lambda)\,\nabla^2(u+fv)\,d\lambda = -4\,(\psi(\lambda)\,v)_{\lambda=\infty}$$

Inspection of the values obtained shows that the limit of $\psi(\lambda)v$ when $\lambda = \infty$ is zero, so that equation (51) reduces to $\nabla^2 V'_i = -4\pi\rho$, and since V'_i is equal to the true value of V_i at infinity, and is finite at the origin, V'_i must be the true value of the internal potential. Thus the potentials are given by,

$$V_i = \int_0^\infty \psi(\lambda)\,[f+e\,(u_1+fv_1)+e^2\,(u_2+fw+f^2w')]\,d\lambda \quad . \quad . \quad . \quad . \quad (52)$$

$$V_0 = \int_\lambda^\infty \psi(\lambda)\,[f+e\,(u_1+fv_1)+e^2\,(u_2+fw+f^2w')]\,d\lambda \quad . \quad . \quad . \quad . \quad (53)$$

in which all the quantities must be transformed into x, y, z co-ordinates before integration.

When $\lambda = 0$, u_2 reduces to $\frac{1}{4}\omega\,(\xi,\eta,\zeta)$ by equation (40),

or to $\frac{1}{4}\omega\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right)$ while $u_1 = \chi\,(\xi,\eta,\zeta) = \chi\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right)$,

also v_1, w and w' all vanish when $\lambda = 0$, so that (*cf.* equation (50))

$$\phi_{\lambda=0} = e\chi\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right) + \tfrac{1}{4}e^2\omega\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right)$$

and the boundary of the distorted ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1+e\chi\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right)+\tfrac{1}{4}e^2\omega\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right) = 0 \quad . \quad . \quad . \quad . \quad (54)$$

16. Before proceeding further it will be convenient to examine in detail the first order solutions which can be obtained from the foregoing analysis, classifying them according to the degree n of the algebraic function u_1, and, for brevity, omitting the continual multiplier e.

n = 0. Solution is $u_1 = \kappa$, $\quad v_1 = 0$, $\quad \phi = \kappa$.

n = 1. Solution is $u_1 = p\xi+q\eta+r\zeta$, $\quad v_1 = 0$, $\quad \phi = \dfrac{px}{A}+\dfrac{qy}{B}+\dfrac{rz}{C}$.

n = 2. $u_1 = \alpha\xi^2+\beta\eta^2+\gamma\zeta^2+2f\eta\zeta+2g\zeta\xi+2h\xi\eta$,

$$v_1 = -\tfrac{1}{4}\int_0^\lambda \nabla^2_{\xi\eta\zeta}u_1\,d\lambda = -\tfrac{1}{2}\lambda\left(\frac{\alpha}{a^2A}+\frac{\beta}{b^2B}+\frac{\gamma}{c^2C}\right),$$

$$\phi = \alpha\left(\frac{x^2}{A^2}-\frac{1}{2}\frac{\lambda}{a^2A}f\right)+\beta\left(\frac{y^2}{B^2}-\frac{1}{2}\frac{\lambda}{b^2B}f\right)+\gamma\left(\frac{z^2}{C^2}-\frac{1}{2}\frac{\lambda}{c^2C}f\right)+2f\eta\zeta+2g\zeta\xi+2h\xi\eta.$$

A physical interpretation of these first few solutions can readily be found. For the undisturbed ellipsoid of axes ka, kb, kc, and origin at x_0, y_0, z_0,

$$\Phi = -\frac{\pi\rho abc}{A^{1/2}B^{1/2}C^{1/2}}\left(\frac{(x-x_0)^2}{A}+\frac{(y-y_0)^2}{B}+\frac{(z-z_0)^2}{C}-k^2\right), \quad . \quad . \quad . \quad . \quad (55)$$

and the special ellipsoid which has been under consideration has been that for which $x_0 = y_0 = z_0 = 0$, $k = 1$. We can change the centre and axes of the ellipsoid contemplated in equation (55) by varying x_0, y_0, z_0, a^2, b^2, c^2, and k. If we change k^2 by an amount δk^2 in equation (55), the change in Φ is given by $\delta\Phi = \frac{\pi\rho abc}{A^{1/2}B^{1/2}C^{1/2}}\,\delta k^2$, so that f may be regarded as replaced by $f+\phi$ where $\phi = -\delta k^2$. Thus the solution $n = 0$ represents a change from k^2 to $k^2-\kappa$; physically it represents a change in the scale of the ellipsoid.

Similarly, if in equation (55) x_0 is changed by δx_0, y_0 by δy_0, and z_0 by δz_0, we find that $\delta\Phi = -2\psi(\lambda)\left(\frac{\delta x_0 x}{A}+\frac{\delta y_0 y}{B}+\frac{\delta z_0 z}{C}\right)$, so that

$$\phi = -2\left(\frac{\delta x_0 x}{A}+\frac{\delta y_0 y}{B}+\frac{\delta z_0 z}{C}\right),$$

and the solution $n = 1$ is seen to represent a motion of the centre of the ellipsoid.

Similarly, if we put $x_0 = y_0 = z_0 = 0$ and $k = 1$ in equation (55) so that $\Phi = \psi f$, and differentiate logarithmically with respect to a^2, we obtain

$$\frac{1}{\psi f}\frac{\partial\Phi}{\partial a^2} = \tfrac{1}{2}\frac{1}{a^2}-\tfrac{1}{2}\frac{1}{A}-\frac{1}{f}\frac{x^2}{(a^2+\lambda)^2},$$

whence

$$\phi = \left(\tfrac{1}{2}\frac{\lambda f}{a^2 A}-\frac{x^2}{A^2}\right)\delta a^2.$$

Clearly, then, the first three terms in the solution $n = 2$ represent a distortion of the original ellipsoid produced by a change in the lengths of the axes, and it is easily seen that the complete solution represents a change of this kind combined with a small rotation of the axes.

n = 3. There are ten terms in the general cubic function of ξ, η, ζ. For the present purpose it is convenient to regard this general cubic function as made up of a term $\epsilon\xi\eta\zeta$, and the sum of three expressions such as

$$\xi\,(\alpha\,\xi^2+\beta\eta^2+\gamma\,\zeta^2).$$

For the solution given by $u_1 = \epsilon\xi\eta\zeta$, we have $\nabla^2_{\xi\eta\zeta}u_1 = 0$, so that $v_1 = 0$ and

$$\phi = \epsilon\frac{xyz}{ABC}.$$

It will be shown later that an ellipsoid distorted in this way cannot possibly be a figure of equilibrium for a rotating fluid (§ 20).

For the solution

$$u_1 = \xi(\alpha\xi^2+\beta\eta^2+\gamma\zeta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

we have

$$v_1 = -\tfrac{1}{4}\int_0^\lambda \nabla^2_{\xi\eta\zeta}\, u_1\, d\lambda = -\xi\left(\frac{3\alpha\lambda}{2a^2\mathrm{A}} + \frac{\beta\lambda}{2b^2\mathrm{B}} + \frac{\gamma\lambda}{2c^2\mathrm{C}}\right)$$

and the solution is

$$\phi = \frac{x}{\mathrm{A}}\left\{\alpha\frac{x^2}{\mathrm{A}^2}+\beta\frac{y^2}{\mathrm{B}^2}+\gamma\frac{z^2}{\mathrm{C}^2}\right\} - \tfrac{1}{2}\left(\frac{x^2}{\mathrm{A}}+\frac{y^2}{\mathrm{B}}+\frac{z^2}{\mathrm{C}}-1\right)\left(\frac{3\alpha\lambda}{a^2\mathrm{A}}+\frac{\beta\lambda}{b^2\mathrm{B}}+\frac{\gamma\lambda}{c^2\mathrm{C}}\right). \quad . \quad (57)$$

It will be shown later that this distortion leads to the Darwin-Poincaré series of pear-shaped figures of equilibrium of a rotating fluid.

n = 4. The analysis of §§ 13, 14 was confined to the case in which u_1 was supposed of degree not greater than 3 in ξ, η, ζ. But if, in the solution finally obtained in § 15, we take $u_1 = 0$, which involves taking also $v_1 = 0$, we are left with a solution (*cf.* equation (50))

$$\phi = e^2(u_2+fw+f^2w'),$$

in which u_2, w, and w' do not vanish on account of the occurrence of the arbitrary function ω. And since ω has been supposed of the fourth degree, this solution gives us the solution of degree $n = 4$ to the first order, the parameter e^2 replacing the former e. Thus the solution of degree 4 is

$$u_1 = \text{a general function of degree 4 in } \xi,\ \eta,\ \zeta \text{ (the old } e^2u_2)$$

$$\begin{aligned}\phi &= u_1+fw+f^2w',\\ &= u_1-\tfrac{1}{4}f\mathrm{D}u_1+\tfrac{1}{64}f^2\mathrm{D}^2u_1.\end{aligned}$$

This solution is not discussed in detail in the present paper, but is classified here with the other solutions for the sake of completeness. An ellipsoid distorted in accordance with this solution would give rise to a series of dumb-bell shaped figures, which would be figures of equilibrium for a rotating liquid. They would be unstable for a homogeneous mass, but the corresponding figures might conceivably be stable for a heterogeneous mass (*cf.* POINCARÉ's remark quoted in § 1 of the present paper).

17. One point of interest must be mentioned here in connection with the potentials derived from these solutions.

In the potentials arising from the solution of degree $n = 2$, $u_1 = \alpha\xi^2$, the internal or boundary potential will be of the form $lx^2+my^2+nz^2$, where l, m, n do not involve x, y, z, or λ. Since this must be a solution of LAPLACE's equation, $l+m+n$ must vanish, and the potential must be expressible in the form $m(y^2-x^2) + n(z^2-x^2)$. All the other potentials may be similarly treated.

Making use of this simplification, we arrive at the following scheme for the contributions to the internal or boundary potentials of the various solutions up to $n = 3$. Only typical terms are taken; ϕ_b represents the value of ϕ at the boundary, V_b represents the contribution from the typical term to the boundary or internal potential.

n = 0. $\phi_b = \kappa$, $V_b = \kappa \int_0^\infty \psi d\lambda.$

n = 1. $\phi_b = \dfrac{x}{a^2}$, $V_b = x \int_0^\infty \dfrac{\psi}{A} d\lambda.$

n = 2. (i) $\phi_b = \dfrac{xy}{a^2 b^2}$, $V_a = xy \int_0^\infty \dfrac{\psi}{AB} d\lambda.$

(ii) $\phi_b = \dfrac{x^2}{a^4}$, $V_b = -\int_0^\infty \dfrac{\psi\lambda}{2a^2 A} \left\{ \dfrac{y^2-x^2}{B} - \dfrac{z^2-x^2}{C} - 1 \right\} d\lambda.$

n = 3. (i) $\phi_b = \dfrac{xyz}{a^2 b^2 c^2}$, $V_b = xyz \int_0^\infty \dfrac{\psi}{ABC} d\lambda.$

(ii) $\phi_b = \dfrac{xy^2}{a^2 b^4}$, $V_b = -\int_0^\infty \dfrac{\psi\lambda x}{2b^2 AB} \left\{ \dfrac{x^2-3y^2}{A} + \dfrac{z^2-y^2}{C} - 1 \right\} d\lambda.$

(iii) $\phi_b = \dfrac{x^3}{a^6}$ $V_b = -\int_0^\infty \dfrac{\psi\lambda x}{2a^2 A^2} \left\{ \dfrac{3y^2-x^2}{B} + \dfrac{3z^2-x^2}{C} - 3 \right\} d\lambda.$

18. In any physical application of this method, and in particular in its application to the discussion of rotating masses of liquid, it will be important to know what changes are produced by the distortion upon the mass (or density), the position of the centre of gravity, and the moments of inertia of the body. These changes are given at once by a study of the limiting form of the external potential at infinity.

The potential at infinity of any mass whatever, taken as far as terms of order $\dfrac{1}{r^3}$, has the limiting form

$$\frac{m}{r} + \frac{m(x_0 x + y_0 y + z_0 z)}{r^3} + \frac{3}{2r^5}(Lx^2 + My^2 + Nz^2 + 2Pyz + 2Qzx + 2Rxy)$$
$$- \frac{1}{2r^3}(L+M+N) + \ldots,$$

where m is the mass of the whole body, x_0, y_0, z_0, are the co-ordinates of the centre of gravity, and, L, M, N, P, Q, R, are products of inertia defined by $\iiint \rho x^2 \, dx \, dy \, dz = L$, $\iiint \rho yz \, dx \, dy \, dz = P$, &c. The moment of inertia about the axis of z is $\iiint \rho \, (x^2+y^2)$ $dx \, dy \, dz = L+M$, and so on.

For the solution $\phi_b = \kappa$ of degree $n = 0$, the limit at infinity of the contribution to the potential is

$$\delta V_\infty = \kappa \int_\lambda^\infty - \frac{\pi\rho abc}{\Delta} d\lambda = - \frac{2\pi\rho abc\kappa}{r} + \ldots,$$

showing that the distortion involves a change of mass $\delta M = -2\pi\rho abc\kappa$, accompanied of course with a change in the inertia terms.

For the solution $\phi_b = \frac{x}{a^2}$ of order $n = 1$,

$$\delta V_\infty = x \int_\lambda^\infty \frac{-\pi\rho abc}{\Delta A} d\lambda = -\tfrac{2}{3}\pi\rho abc \frac{a^2 x}{r^3} = -\tfrac{1}{2}a^2 \frac{mx}{r^3},$$

so that this distortion represents a motion of the centre of gravity by an amount $\delta x_0 = -\frac{1}{2}a^2$.

Solutions of degree $n = 2$ will clearly involve changes in the moments and products of inertia. The limiting potentials are found to be as follows:

$$\text{(i)} \quad \phi_b = \frac{xy}{a^2 b^2}, \qquad \delta V_\infty = -\tfrac{2}{5}\pi\rho abc \frac{xy}{r^5},$$

$$\text{(ii)} \quad \phi_b = \frac{x^2}{a^4}, \qquad \delta V_\infty = -\tfrac{2}{5}\pi\rho abc \frac{x^2}{r^5} - \tfrac{2}{3}\pi\rho abc \frac{1}{a^2 r}.$$

The first solution does not involve a change in mass, whilst the second does; both distortions affect the inertia.

For the solutions of degree $n = 3$, the limiting values are as follows:

$$\text{(i)} \quad \phi_b = \frac{xyz}{a^2 b^2 c^2}, \qquad \delta V_\infty = -\tfrac{2}{7}\pi\rho abc \frac{xyz}{r^7}.$$

This distortion changes neither mass, centre of gravity, nor inertia.

$$\text{(ii)} \quad \phi_b = \frac{xy^2}{a^2 b^4}, \qquad \delta V_\infty = -\tfrac{2}{7}\pi\rho abc \frac{xy^2}{r^7} - \tfrac{2}{15}\pi\rho abc \frac{x}{b^2 r^3},$$

$$\text{(iii)} \quad \phi_b = \frac{x^3}{a^6}, \qquad \delta V_\infty = -\tfrac{2}{7}\pi\rho abc \frac{x^3}{r^7} - \tfrac{2}{5}\pi\rho abc \frac{x}{a^2 r^3}.$$

These two latter distortions move the centre of gravity, but do not affect the mass or inertia.

It is clear, without detailed examination, that the distortions represented by solutions of degree $n = 4$ cannot affect the centre of gravity, but may affect the mass and inertia.

Figure of Equilibrium of Rotating Masses.

The Jacobian Ellipsoids.

19. The condition that a single figure shall be a figure of equilibrium for a rotation ω about the axis of z is that the centre of gravity shall lie on the axis of z, and that $V_b+\frac{1}{2}\omega^2(x^2+y^2)$ shall have a constant value over the boundary $\lambda = 0$. In searching for a series of figures of equilibrium we must add a further condition of constancy of mass.

For the undisturbed ellipsoid, with the notation of § 10, $V_b+\frac{1}{2}\omega^2(x^2+y^2)$

$$= x^2\left\{-\int_0^\infty \frac{\pi\rho abc}{\Delta A}\,d\lambda+\tfrac{1}{2}\omega^2\right\} + y^2\left\{-\int_0^\infty \frac{\pi\rho abc}{\Delta B}\,d\lambda+\tfrac{1}{2}\omega^2\right\} + z^2\left\{-\int_0^\infty \frac{\pi\rho abc}{\Delta C}\,d\lambda\right\}. \quad (58)$$

For this to be constant over the surface, it must be identical with

$$\chi\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right),$$

where χ is a constant. If we put

$$\frac{\omega^2}{2\pi\rho abc} = n, \qquad \frac{\chi}{\pi\rho abc} = \theta,$$

the equations obtained by comparing coefficients are

$$\int_0^\infty \frac{d\lambda}{\Delta A} - n = \frac{\theta}{a^2}, \quad (59)$$

$$\int_0^\infty \frac{d\lambda}{\Delta B} - n = \frac{\theta}{b^2}, \quad (60)$$

$$\int_0^\infty \frac{d\lambda}{\Delta C} = \frac{\theta}{c^2}. \quad (61)$$

Ellipsoids with Distortions of the First Order.

20. We proceed to consider which of the distorted ellipsoids can give rise to possible figures of equilibrium.

The solutions of degrees 0, 1, 2 lead to nothing except new ellipsoids, so that the inclusion of these distortions could only represent a step along the already known series of Jacobian ellipsoids or Maclaurin spheroids.

Consider next an ellipsoid distorted by the addition of a solution of the type (i) of degree 3 (§ 18), say $\phi_b = \epsilon\frac{xyz}{a^2b^2c^2}$. This distortion, as we have seen (§ 18), does not affect the total mass or the position of the centre of gravity. The boundary of the distorted ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1+\epsilon\frac{xyz}{a^2b^2c^2};$$

while the additional term which has to be inserted on the right of equation (58) is

$$-\epsilon xyz\int_0^\infty \frac{\pi\rho abc}{\Delta\mathrm{ABC}}\,d\lambda.$$

Hence the additional equation which has to be satisfied, in addition to equations (59) to (61), is

$$\int_0^\infty \frac{d\lambda}{\Delta\mathrm{ABC}} = \frac{\theta}{a^2b^2c^2}. \quad \ldots \ldots \quad (62)$$

Eliminating θ from this equation and (61) we obtain, as an equation which must be satisfied if the distorted ellipsoid is to be a figure of equilibrium,

$$\int_0^\infty \frac{d\lambda}{\Delta\mathrm{C}}\left(1-\frac{a^2b^2}{(a^2+\lambda)(b^2+\lambda)}\right) = 0. \quad \ldots \ldots \quad (63)$$

This obviously cannot be satisfied, for the integrand is positive for all values of λ. We conclude that the distortion now under consideration cannot possibly give rise to a figure of equilibrium.

21. There remain nine terms for consideration in the general cubic function. Inspection will show, or it will soon become apparent as we proceed with the analysis, that these fall into three groups, as in § 16, and that the three terms of any one group just suffice to give a possible figure of equilibrium when combined with a term to restore the centre of gravity to its position on the axis of rotation. We shall accordingly consider a distortion in which the cubic terms are those already written down in equation (56). These terms are seen (§ 18) to move the centre of gravity parallel to the axis of x, and to correct this we shall add a term (*cf.* § 16), $\frac{\kappa x}{\mathrm{A}}$ to u_1.

Thus, for the distorted ellipsoid now under consideration, the boundary will be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1+e\left(\alpha\frac{x^3}{a^6}+\beta\frac{xy^2}{a^2b^4}+\gamma\frac{xz^2}{a^2c^4}+\kappa\frac{x}{a^2}\right). \quad \ldots \ldots \quad (64)$$

As far as terms in $\frac{1}{r^2}$, the value of the potential at infinity is (*cf.* § 18)

$$-\pi\rho abce\frac{x}{r^3}\left\{\tfrac{2}{5}\frac{\alpha}{a^2}+\tfrac{2}{15}\frac{\beta}{b^2}+\tfrac{2}{15}\frac{\gamma}{c^2}+\tfrac{2}{3}\kappa\right\},$$

so that for the centre of gravity to remain at the origin we must have

$$\kappa = -\tfrac{1}{5}\left(\frac{3\alpha}{a^2}+\frac{\beta}{b^2}+\frac{\gamma}{c^2}\right). \quad \ldots \ldots \quad (65)$$

Collecting the terms in V_b as calculated in § 18, we find

$$
\begin{aligned}
&-\left[V_b+\tfrac{1}{2}\omega^2(x^2+y^2)\right] \\
&= x^2\left\{\int_0^\infty \frac{\pi\rho abc}{\Delta A}\,d\lambda-\tfrac{1}{2}\omega^2\right\}+y^2\left\{\int_0^\infty \frac{\pi\rho abc}{\Delta B}\,d\lambda-\tfrac{1}{2}\omega^2\right\}+z^2\int_0^\infty \frac{\pi\rho abc}{\Delta C}\,d\lambda \\
&\quad + e\,\frac{\alpha x}{2a^2}\int_0^\infty \frac{\psi\lambda}{A^2}\left\{\frac{3y^2-x^2}{B}+\frac{3z^2-x^2}{C}-3\right\}d\lambda \\
&\quad + e\,\frac{\beta x}{2b^2}\int_0^\infty \frac{\psi\lambda}{AB}\left\{\frac{x^2-3y^2}{A}+\frac{z^2-y^2}{C}-1\right\}d\lambda \\
&\quad + e\,\frac{\gamma x}{2c^2}\int_0^\infty \frac{\psi\lambda}{AC}\left\{\frac{x^2-3z^2}{A}+\frac{y^2-z^2}{B}-1\right\}d\lambda-e\kappa x\int_0^\infty \frac{\psi}{A}\,d\lambda. \quad . \quad . \quad . \quad . \quad (66)
\end{aligned}
$$

For brevity in printing, introduce the following notation. Let

$$\int_0^\infty \frac{d\lambda}{\Delta ABC\ldots}=J_{ABC\ldots}\,; \qquad \int_0^\infty \frac{\lambda\,d\lambda}{\Delta ABC\ldots}=I_{ABC\ldots}, \quad . \quad . \quad . \quad . \quad . \quad (67)$$

so that, for instance,

$$J_A=\int_0^\infty \frac{d\lambda}{\Delta A}, \qquad I_{AA}=\int_0^\infty \frac{d\lambda}{\Delta A^2}.$$

And, for the problem immediately in hand, write further

$$c_1=I_{ABC}=\int_0^\infty \frac{d\lambda}{\Delta ABC}, \qquad c_2=I_{AAC}, \qquad c_3=I_{AAB},$$

and as before put $\dfrac{\omega^2}{2\pi\rho abc}=n$, then equation (66) becomes

$$
\begin{aligned}
&-\left[V_b+\tfrac{1}{2}\omega^2(x^2+y^2)\right] \\
&= \pi\rho abc\Big[x^2(J_A-n)+y^2(J_B-n)+z^2J_C \\
&\qquad + e\,\frac{\alpha x}{2a^2}\left\{x^2(c_2+c_3)-3y^2c_3-3z^2c_2\right\} \\
&\qquad + e\,\frac{\beta x}{2b^2}\left\{-x^2c_3+y^2(3c_3+c_1)-z^2c_1\right\} \\
&\qquad + e\,\frac{\gamma x}{2c^2}\left\{-x^2c_2-y^2c_1+z^2(3c_2+c_1)\right\} \\
&\qquad + ex\left\{\frac{3\alpha}{2a^2}I_{AA}+\frac{\beta}{2b^2}I_{AB}+\frac{\gamma}{2c^2}I_{AC}+\kappa J_A\right\}\Big]. \quad . \quad . \quad . \quad (68)
\end{aligned}
$$

The distorted ellipsoid will be a possible figure of equilibrium, as far as the first order of small quantities, provided the right-hand member of equation (68) is identical with

$$\pi\rho abc\theta\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1+ex\left(\alpha\frac{x^2}{a^6}+\beta\frac{y^2}{a^2b^4}+\gamma\frac{z^2}{a^2c^4}+\frac{\kappa}{a^2}\right)\right]. \quad . \quad . \quad . \quad (69)$$

Equating coefficients, we obtain

$$\mathrm{J_A}-n=\frac{\theta}{a^2}, \qquad \mathrm{J_B}-n=\frac{\theta}{b^2}, \qquad \mathrm{J_C}=\frac{\theta}{c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

$$\frac{\alpha}{2a^2}(c_2+c_3)-\frac{\beta}{2b^2}c_3-\frac{\gamma}{2c^2}c_2=\theta\frac{\alpha}{a^6}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

$$-\frac{3\alpha}{2a^2}c_3+\frac{\beta}{2b^2}(3c_3+c_1)-\frac{\gamma}{2c^2}c_1=\theta\frac{\beta}{a^2b^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

$$-\frac{3\alpha}{2a^2}c_2-\frac{\beta}{2b^2}c_1+\frac{\gamma}{2c^2}(3c_2+c_1)=\theta\frac{\gamma}{a^2c^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

$$\frac{3\alpha}{2a^2}\mathrm{I_{AA}}+\frac{\beta}{2b^2}\mathrm{I_{AB}}+\frac{\gamma}{2c^2}\mathrm{I_{AC}}+\kappa\,\mathrm{J_A}=\theta\frac{\kappa}{a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

and from these equations, together with equation (65), we must eliminate or obtain the coefficients.

If we put

$$\frac{\alpha}{a^2}=\alpha', \qquad \frac{\beta}{b^2}=\beta', \qquad \frac{\gamma}{c^2}=\gamma' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

then equations (71) to (73) reduce to

$$\alpha'(c_2+c_3)-\beta'c_3-\gamma'c_2=\frac{2\theta}{a^4}\alpha', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

$$\alpha'(-3c_3)+\beta'(3c_3+c_1)-\gamma'c_1=\frac{2\theta}{a^2b^2}\beta', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

$$\alpha'(-3c_2)-\beta'c_1+\gamma'(3c_2+c_1)=\frac{2\theta}{a^2c^2}\gamma', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

whilst on substitution for κ from equation (65) and $\mathrm{J_A}$ from (70), equation (74) reduces to

$$\tfrac{1}{2}a^2(3\alpha'\mathrm{I_{AA}}+\beta'\mathrm{I_{AB}}+\gamma'\mathrm{I_{AC}})=\tfrac{1}{5}na^2(3\alpha'+\beta'+\gamma'). \quad . \quad . \quad . \quad . \quad . \quad (79)$$

We have now to deal with four equations (76), (77), (78), and (79), and have to examine whether, and how, these can be satisfied by values of α', β', and γ'. Since there must, from general physical principles, be an equation of some sort for points of bifurcation (whether capable of being satisfied by real values or not), we are led to suspect that these four equations are not really independent.

Equation (79) was obtained by the elimination of κ from two equations (65) and (74), each of which expressed in effect the condition that the centre of gravity of the mass should be at the origin; in fact, equation (65) was only a short way of arriving at the value of κ, which would in any case have been given by equation (74). We therefore expect that equation (79), derived from (65) and (74), will prove only to be an identity of which the truth is involved in the three other equations (76) to (78). And, as a matter of procedure which is entirely at our choice, we shall elect first to solve equations (76) to (78), and then to verify the truth of (79).

The elimination of α', β', γ' from equations (76), (77), and (78) gives a determinant which on expansion reduces to

$$a^2b^2c^2\left(\frac{3}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(c_1c_2+c_1c_3+3c_2c_3)$$
$$-\frac{2\theta}{a^2}\{c_1(b^2+c^2)+c_2(3a^2+c^2)+c_3(3a^2+b^2)\}+\left(\frac{2\theta}{a^2}\right)^2=0, \quad (80)$$

and this is accordingly the equation giving points of bifurcation on the Jacobian series of ellipsoids.

22. Two identities of importance are the following:

$$J_A+J_B+J_C=\int_0^\infty\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)\frac{d\lambda}{\Delta}=-2\int_0^\infty\frac{\partial}{\partial\lambda}\left(\frac{1}{\Delta}\right)d\lambda=\frac{2}{abc}, \quad . \quad . \quad . \quad (81)$$

$$3I_{AA}+I_{AB}+I_{AC}=\int_0^\infty\frac{\lambda}{\Delta A}\left(\frac{3}{A}+\frac{1}{B}+\frac{1}{C}\right)d\lambda=-2\int_0^\infty\lambda\frac{\partial}{\partial\lambda}\left(\frac{1}{\Delta A}\right)d\lambda=2J_A. \quad (82)$$

From equation (70)

$$J_A=n+\frac{\theta}{a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

$$J_B=n+\frac{\theta}{b^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

$$J_C=\frac{\theta}{c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

so that on addition, by the use of (81),

$$\frac{2}{abc}=2n+\theta\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (86)$$

giving θ, and hence J_A, J_B, and J_C, in terms of a, b, c, and n. We further have

$$I_{AB}=\int_0^\infty\frac{\lambda\,d\lambda}{\Delta AB}=\int_0^\infty\left\{\frac{a^2}{(a^2-b^2)\Delta A}-\frac{b^2}{(a^2-b^2)\Delta B}\right\}d\lambda=\frac{a^2J_A-b^2J_B}{a^2-b^2} \quad . \quad . \quad (87)$$

$$I_{AC}=\frac{a^2J_A-c^2J_C}{a^2-c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

and I_{AA} is given by equation (80) as soon as the values of I_{AB} and I_{AC} are known. Substituting for J_A, J_B, J_C, from equations (83) to (85), we obtain

$$I_{AB} = n, \qquad I_{AC} = \frac{a^2}{a^2-c^2}n, \qquad I_{AA} = -\tfrac{1}{3}\left\{\frac{c^2}{a^2-c^2}n-\frac{2\theta}{a^2}\right\}. \quad . \quad . \quad (89)$$

Finally we have

$$c_1 = \int_0^\infty \frac{\lambda\, d\lambda}{\Delta ABC} = \frac{1}{c^2-b^2}\int_0^\infty \left(\frac{1}{B}-\frac{1}{C}\right)\frac{\lambda\, d\lambda}{\Delta A} = \frac{1}{(c^2-b^2)}(I_{AB}-I_{AC}),$$

$$c_2 = \frac{1}{c^2-a^2}(I_{AA}-I_{AC}), \qquad c_3 = \frac{1}{a^2-b^2}(I_{AB}-I_{AA}).$$

With this material it ought to be possible to find the points of bifurcation from equation (80). As, however, DARWIN's results are available, it will be sufficient to make use of his results and merely verify that his quantities satisfy equation (80), as they are in point of fact found to do.

23. DARWIN's values, calculated for an ellipsoid such that $abc = 1$, are

$$a = 1{\cdot}885827, \qquad b = 0{\cdot}814975, \qquad c = 0{\cdot}650659,$$

$$n = \frac{\omega^2}{2\pi\rho} = 0{\cdot}1419990,$$

whence, by equations (86) and (89),

$$\tfrac{1}{2}\theta = \frac{1-n}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} = 0{\cdot}2068037$$

$$I_{AB} = 0{\cdot}1419990, \qquad I_{AC} = 0{\cdot}1611871, \qquad I_{AA} = 0{\cdot}0711382,$$

$$c_1 = 0{\cdot}07967602, \qquad c_2 = 0{\cdot}02874219, \qquad c_3 = 0{\cdot}2450100.$$

With the use of these values, equations (76), (77) and (78) become

$$0{\cdot}01216184\alpha'+0{\cdot}02450100\beta'+0{\cdot}02874219\gamma' = 0 \quad . \quad . \quad . \quad . \quad . \quad (90)$$

$$0{\cdot}07350300\alpha'+0{\cdot}1970290\beta'+0{\cdot}07967602\gamma' = 0 \quad . \quad . \quad . \quad . \quad . \quad (91)$$

$$0{\cdot}0862266\alpha'+0{\cdot}07967602\beta'+0{\cdot}3835217\gamma' = 0 \quad . \quad . \quad . \quad . \quad . \quad (92)$$

The values of α', β', γ', are, of course, indeterminate to within a common multiplier. The simplest set of values, obtained by cross multiplication of the coefficients of equations (90) and (91), is

$$\alpha' = -0{\cdot}003710945, \qquad \beta' = 0{\cdot}001143630 \qquad \gamma' = 0{\cdot}000595338.$$

If we substitute these values in equation (92), we find

$$0{\cdot}0862266\alpha' + 0{\cdot}07967602\beta' + 0{\cdot}3835217\gamma'$$
$$= -0{\cdot}00031998 + 0{\cdot}00009111 + 0{\cdot}00022829 = -0{\cdot}0000005.$$

The fact that the error occurs only in the seventh place of decimals adequately verifies DARWIN'S calculations, but the tendency of small errors to accumulate in computation is forcibly illustrated by the circumstance that in the above equation the final error is as much as one-six-hundredth part of the whole value of the middle term.

With the values just obtained for α', β', γ', I find

$$\tfrac{1}{5}a^2 (3\alpha' + \beta' + \gamma')\, n = -0{\cdot}00094878,$$
$$\tfrac{1}{2}a^2 (3\alpha'\, \mathrm{I}_{\mathrm{AA}} + \beta'\, \mathrm{I}_{\mathrm{AB}} + \gamma'\, \mathrm{I}_{\mathrm{AC}}) = -0{\cdot}00094885,$$

verifying that equation (79) is satisfied, again as far as the sixth place of decimals.

24. With a view to subsequent computations, it is convenient to take a standard set of values such that $\alpha' = -1$. These values are found to be

$$\alpha' = -1, \qquad \beta' = 0{\cdot}3081810, \qquad \gamma' = 0{\cdot}1604294,$$

and with these we have, by equation (65),

$$\kappa = -\tfrac{1}{5}(3\alpha' + \beta' + \gamma') = 0{\cdot}506278.$$

These numerical values substituted in equation (64) will give the equation of POINCARÉ'S pear-shaped figure as far as small terms of the first order.

The Pear-shaped Figure Calculated to the Second Order.

25. The question as to whether the pear-shaped figure is stable depends upon the change effected by the distortion upon the angular momentum of the ellipsoid. But (*cf.* § 18) the first-order distortion so far considered can be easily seen to produce no effect at all upon the angular momentum of the figure. It is therefore necessary to proceed to terms of a higher order, and we now consider terms of the second order.

The first-order terms have been found to be given by

$$u_1 = \xi(\alpha\xi^2 + \beta\eta^2 + \gamma\zeta^2 + \kappa), \quad \ldots \ldots \quad (93)$$

with (*cf.* equation (47)) $v_1 = -\frac{1}{4}\mathrm{D}u_1$. The value of u_2 will be given by equation (48), in which u_1 is to be assigned the value (93), and ω will be taken to be given by

$$\omega = \mathrm{L}\xi^4 + \mathrm{M}\eta^4 + \mathrm{N}\zeta^4 + 2l\eta^2\zeta^2 + 2m\zeta^2\xi^2 + 2n\xi^2\eta^2 + 2(p\xi^2 + q\eta^2 + r\zeta^2) + s. \quad . \quad . \quad (94)$$

It has to be shown that this value for ω makes it possible for the figure distorted to the second order in this way to be a figure of equilibrium.

It will be noticed that with the value we have assumed for ω, the value of u_2 becomes a function of ξ, η, ζ, of degree 4 involving only even powers of ξ, η, ζ; its value is

$$4u_2 = 2\xi^2(\alpha\xi^2+\beta\eta^2+\gamma\zeta^2+\kappa)\left(3\alpha\left(\frac{1}{\mathrm{A}}-\frac{1}{a^2}\right)+\beta\left(\frac{1}{\mathrm{B}}-\frac{1}{b^2}\right)+\gamma\left(\frac{1}{\mathrm{C}}-\frac{1}{c^2}\right)\right)$$
$$+\left(\frac{1}{\mathrm{A}}-\frac{1}{a^2}\right)(3\alpha\xi^2+\beta\eta^2+\gamma\zeta^2+\kappa)^2+4\left(\frac{1}{\mathrm{B}}-\frac{1}{b^2}\right)\beta^2\xi^2\zeta^2+4\left(\frac{1}{\mathrm{C}}-\frac{1}{c^2}\right)\gamma^2\xi^2\zeta^2$$
$$+\mathrm{L}\xi^4+\mathrm{M}\eta^4+\mathrm{N}\zeta^4+2l\eta^2\zeta^2+2m\zeta^2\xi^2+2n\xi^2\eta^2+2(p\xi^2+q\eta^2+r\zeta^2)+s. \quad . \quad . \quad (95)$$

The values of fw and of f^2w' are easily seen to be similar in form, so that all the second-order terms in ϕ are of this form (*cf.* equation (50)). Multiplying by $\psi(\lambda)\,d\lambda$ and integrating from 0 to ∞, we obtain as the terms of the second order in the potential an expression of the form

$$-\pi\rho abce^2(c_{11}x^4+c_{22}y^4+c_{33}z^4+c_{12}x^2y^2+c_{23}y^2z^2+c_{31}z^2x^2+d_1x^2+d_2y^2+d_3z^2+d_4).$$

If this figure can be a figure of equilibrium at all, it will be for a rotation differing only by a second-order quantity from that of the original ellipsoid. Let us suppose that for it $\dfrac{\omega^2}{2\pi\rho abc} = n+e^2n''$; then at the boundary, as far as e^2,

$$-[\mathrm{V}_b+\tfrac{1}{2}\omega^2(x^2+y^2)]$$
$$= \pi\rho abc\,\{x^2(\mathrm{J_A}-n-e^2n'')+y^2(\mathrm{J_B}-n-e^2n'')+z^2\mathrm{J_C}$$
$$+e^2(c_{11}x^4+c_{22}y^4+c_{33}z^4+c_{12}x^2y^2+c_{23}y^2z^2+c_{31}z^2x^2+d_1x^2+d_2y^2+d_3z^2+d_4$$
$$+ \text{ third-degree terms in } e, \text{ the same as before}\}. \quad . \quad . \quad . \quad . \quad . \quad (96)$$

At the boundary,

$$\phi = eu_1+e^2u_2$$
$$= eu_1+\tfrac{1}{4}e^2\left(\mathrm{L}\frac{x^4}{a^8}+\mathrm{M}\frac{y^4}{b^8}+\mathrm{N}\frac{z^4}{c^8}+2l\frac{y^2z^2}{b^4c^4}+2m\frac{z^2x^2}{c^4a^4}+2n\frac{x^2y^2}{a^4b^4}+2p\frac{x^2}{a^4}+2q\frac{y^2}{b^4}+2r\frac{z^2}{c^4}+s\right), \quad (97)$$

so that for the figure to be one of equilibrium, the right-hand member of equation (96) must be identical with

$$\pi\rho abc\theta\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1+ex\left(\alpha\frac{x^2}{a^6}+\beta\frac{y^2}{a^2b^4}+\gamma\frac{z^2}{a^2c^4}+\frac{\kappa}{a^2}\right)\right.$$
$$+\tfrac{1}{4}e^2\left(\mathrm{L}\frac{x^4}{a^8}+\mathrm{M}\frac{y^4}{b^8}+\mathrm{N}\frac{z^4}{c^8}+2l\frac{y^2z^2}{b^4c^4}+2m\frac{z^2x^2}{c^4a^4}+2n\frac{x^2y^2}{a^4b^4}\right.$$
$$\left.\left.+2p\frac{x^2}{a^4}+2q\frac{y^2}{b^4}+2r\frac{z^2}{c^4}+s\right)\right\}. \quad . \quad (98)$$

Upon equating coefficients of the terms of degree 1 and 3 in x, y, z, we obtain exactly the same equations as were obtained before when the terms in e^2 were omitted. Thus no alteration is produced in the cubic terms until the solution is carried as far as e^3.

The equations obtained by comparison of the coefficients of x^2, y^2, and z^2 are :—

$$J_A - n - e^2 n'' + e^2 d_1 = \theta\left(\frac{1}{a^2} + \tfrac{1}{2} p \frac{e^2}{a^4}\right),$$

and two similar equations. In these the terms in e^2 and the terms independent of e must be equal separately. The latter terms again give equations (70), while the terms in e^2 give

$$d_1 - n'' = \tfrac{1}{2}\frac{\theta p}{a^4} \quad \text{(99)}$$

$$d_2 - n'' = \tfrac{1}{2}\frac{\theta q}{b^4} \quad \text{(100)}$$

$$d_3 = \tfrac{1}{2}\frac{\theta r}{c^4} \quad \text{(101)}$$

Finally, upon equating terms of the fourth degree, we obtain the six equations :

$$c_{11} = \tfrac{1}{4}\frac{\theta L}{a^8}; \qquad c_{22} = \tfrac{1}{4}\frac{\theta M}{b^8}; \qquad c_{33} = \tfrac{1}{4}\frac{\theta N}{c^8}. \quad \text{(102) to (104)}$$

$$c_{23} = \tfrac{1}{2}\frac{\theta l}{b^4 c^4}; \qquad c_{31} = \tfrac{1}{2}\frac{\theta m}{c^4 a^4}; \qquad c_{12} = \tfrac{1}{2}\frac{\theta n}{a^4 b^4}. \quad \text{(105) to (107)}$$

We have seen that equations (70) to (73) must still be true, so that $a, b, c, \alpha, \beta, \gamma, \kappa$ will be the same as before.

26. At present there are eleven quantities to be determined or eliminated, namely, L, M, N, l, m, n, p, q, r, s, and n''. The equations giving these quantities can be simplified in the following way.

By an argument already used in § 17, it appears that the terms in e^2 on the right hand of equation (96) must be harmonic. Thus we must have

$$6c_{11} + c_{12} + c_{13} = 0. \quad \text{(108)}$$

$$6c_{22} + c_{23} + c_{21} = 0. \quad \text{(109)}$$

$$6c_{33} + c_{31} + c_{32} = 0. \quad \text{(110)}$$

$$d_1 + d_2 + d_3 = 0. \quad \text{(111)}$$

These are, of course, not new equations to be satisfied; they reduce to identities when the calculated values are inserted for c_{11}, c_{12}, &c.

With the help of (111), equations (99) to (101) may be replaced by

$$n'' = -\tfrac{1}{4}\theta\left(\frac{p}{a^4}+\frac{q}{b^4}+\frac{r}{c^4}\right), \qquad (112)$$

$$d_1-d_2 = \tfrac{1}{2}\theta\left(\frac{p}{a^4}-\frac{q}{b^4}\right), \qquad (113)$$

$$d_3 = \tfrac{1}{2}\theta\frac{r}{c^4}, \qquad (114)$$

while from equations (102) to (110) we obtain

$$\frac{3\mathrm{L}}{a^4}+\frac{m}{c^4}+\frac{n}{b^4} = 0, \qquad (115)$$

$$\frac{3\mathrm{M}}{b^4}+\frac{l}{c^4}+\frac{n}{a^4} = 0, \qquad (116)$$

$$\frac{3\mathrm{N}}{c^4}+\frac{l}{b^4}+\frac{m}{a^4} = 0. \qquad (117)$$

These three equations with equations (102) to (104), namely,

$$c_{11} = \tfrac{1}{4}\frac{\theta\mathrm{L}}{a^8}, \qquad (118)$$

$$c_{22} = \tfrac{1}{4}\frac{\theta\mathrm{M}}{b^8}, \qquad (119)$$

$$c_{33} = \tfrac{1}{4}\frac{\theta\mathrm{N}}{c^8}, \qquad (120)$$

may be used to replace the group (102) to (107).

27. We proceed to evaluate these quantities in detail. The values of u_1 and v_1 from equation (47) are

$$u_1 = \alpha\xi^3+\beta\xi\eta^2+\gamma\xi\zeta^2+\kappa\xi, \qquad (121)$$

$$v_1 = -\tfrac{1}{2}\xi\lambda\left(\frac{3\alpha}{a^2\mathrm{A}}+\frac{\beta}{b^2\mathrm{B}}+\frac{\gamma}{c^2\mathrm{C}}\right), \qquad (122)$$

and the value of u_2 is already given by equation (95).

For convenience in computation we shall combine all the terms in u_2 which are independent of λ in the first two lines, with the similar terms in the last line; we accordingly write

$$\begin{aligned}
&-2\xi^2(\alpha\xi^2+\beta\eta^2+\gamma\zeta^2+\kappa)\left(\frac{3\alpha}{a^2}+\frac{\beta}{b^2}+\frac{\gamma}{c^2}\right)\\
&-\frac{1}{a^2}(3\alpha\xi^2+\beta\eta^2+\gamma\zeta^2+\kappa)^2-\frac{4}{b^2}\beta^2\xi^2\eta^2-\frac{4}{c^2}\gamma^2\xi^2\zeta^2\\
&+\mathrm{L}\xi^4+\mathrm{M}\eta^4+\mathrm{N}\zeta^4+2l\eta^2\zeta^2+2m\zeta^2\xi^2+2n\xi^2\eta^2+2(p\xi^2+q\eta^2+r\zeta^2)+s\\
&= \mathrm{L}'\xi^4+\mathrm{M}'\eta^4+\mathrm{N}'\zeta^4+2l'\eta^2\zeta^2+2m'\zeta^2\xi^2+2n'\xi^2\eta^2+2(p'\xi^2+q'\eta^2+r'\zeta^2)+s'\\
&= \omega'(\xi,\eta,\zeta). \qquad (123)
\end{aligned}$$

The value of u_2 is now given by

$$4u_2 = 4\xi^2\left(\frac{3\alpha}{2\mathrm{A}}+\frac{\beta}{2\mathrm{B}}+\frac{\gamma}{2\mathrm{C}}\right)(\alpha\xi^2+\beta\eta^2+\gamma\zeta^2+\kappa)$$

$$+\frac{1}{\mathrm{A}}(3\alpha\xi^2+\beta\eta^2+\gamma\zeta^2+\kappa)^2+\frac{4\beta\xi^2\eta^2}{\mathrm{B}}+\frac{4\gamma^2\xi^2\zeta^2}{\mathrm{C}}+\omega'(\xi,\eta,\zeta,) \quad . \quad . \quad (124)$$

whence, by equation (45),

$$4w = -\int_0^\lambda \nabla^2_{\xi,\eta,\zeta}\, u_2\, d\lambda$$

$$= \tfrac{45}{2}\frac{\alpha^2\xi^2}{\mathrm{A}^2}+\beta^2\left(\tfrac{3}{2}\frac{\xi^2}{\mathrm{B}^2}+3\frac{\eta^2}{\mathrm{AB}}\right)+\gamma^2\left(\tfrac{3}{2}\frac{\xi^2}{\mathrm{C}^2}+\frac{3\zeta^2}{\mathrm{AC}}\right)$$

$$+\alpha\beta\left(\frac{6\xi^2}{\mathrm{AB}}+\frac{3\eta^2}{\mathrm{A}^2}\right)+\alpha\gamma\left(\frac{6\xi^2}{\mathrm{AC}}+\frac{3\zeta^2}{\mathrm{A}^2}\right)+\beta\gamma\left(\frac{\xi^2}{\mathrm{BC}}+\frac{\eta^2}{\mathrm{AC}}+\frac{\zeta^2}{\mathrm{AB}}\right)$$

$$+\frac{\kappa}{\mathrm{A}}\left(\frac{3\alpha}{\mathrm{A}}+\frac{\beta}{\mathrm{B}}+\frac{\gamma}{\mathrm{C}}\right)+\Sigma\frac{1}{\mathrm{A}}(3\mathrm{L}'\xi^2+m'\zeta^2+n'\eta^2+p')$$

$$+\mathrm{K}_1\xi^2+\mathrm{K}_2\eta^2+\mathrm{K}_3\zeta^2+\mathrm{K}_4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

where the K's are constants, to be chosen so as to make w vanish when $\lambda = 0$.

The value of $\nabla^2_{\xi,\eta,\zeta}(4w)$ is derived from that of $4w$ by replacing ξ^2, η^2, ζ^2 by $\frac{2}{\mathrm{A}^2}, \frac{2}{\mathrm{B}^2}, \frac{2}{\mathrm{C}^2}$ and omitting all other terms. Thus we have, again from equation (45),

$$4w' = -\tfrac{1}{8}\int_0^\lambda \nabla^2_{\xi,\eta,\zeta}(4w)\, d\lambda$$

$$= \tfrac{1}{4}\left\{\tfrac{15}{2}\frac{\alpha^2}{\mathrm{A}^3}+\frac{3\beta^2}{2\mathrm{AB}^2}+\frac{3\gamma^2}{2\mathrm{AC}^2}+\frac{3\alpha\beta}{\mathrm{A}^2\mathrm{B}}+\frac{3\alpha\gamma}{\mathrm{A}^2\mathrm{C}}+\frac{\beta\gamma}{\mathrm{ABC}}\right.$$

$$\left.+\Sigma\tfrac{3}{2}\frac{\mathrm{L}'}{\mathrm{A}^2}+\Sigma\frac{l'}{\mathrm{BC}}+\frac{\mathrm{K}_1}{\mathrm{A}}+\frac{\mathrm{K}_2}{\mathrm{B}}+\frac{\mathrm{K}_3}{\mathrm{C}}+\mathrm{K}_5\right\}, \quad . \quad . \quad . \quad . \quad (126)$$

in which K_5 is a new constant, chosen so as to make w' vanish when $\lambda = 0$.

On collecting terms, we obtain

$$4(w+fw') = \mathrm{P}_1\xi^2+\mathrm{P}_2\eta^2+\mathrm{P}_3\zeta^2+\mathrm{P}_4 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

where

$$\mathrm{P}_1 = 24\tfrac{3}{8}\frac{\alpha^2}{\mathrm{A}^2}+1\tfrac{7}{8}\frac{\beta^2}{\mathrm{B}^2}+1\tfrac{7}{8}\frac{\gamma^2}{\mathrm{C}^2}+6\tfrac{3}{4}\frac{\alpha\beta}{\mathrm{AB}}+6\tfrac{3}{4}\frac{\alpha\gamma}{\mathrm{AC}}+1\tfrac{1}{4}\frac{\beta\gamma}{\mathrm{BC}}$$

$$+3\tfrac{3}{8}\frac{\mathrm{L}'}{\mathrm{A}}+\tfrac{3}{8}\frac{\mathrm{M}'\mathrm{A}}{\mathrm{B}^2}+\tfrac{3}{8}\frac{\mathrm{N}'\mathrm{A}}{\mathrm{C}^2}+\tfrac{1}{4}\frac{l'\mathrm{A}}{\mathrm{BC}}+1\tfrac{1}{4}\frac{m'}{\mathrm{C}}+1\tfrac{1}{4}\frac{n'}{\mathrm{B}}$$

$$+1\tfrac{1}{4}\mathrm{K}_1+\tfrac{1}{4}\frac{\mathrm{AK}_2}{\mathrm{B}}+\tfrac{1}{4}\frac{\mathrm{AK}_3}{\mathrm{C}}+\tfrac{1}{4}\mathrm{K}_5\mathrm{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

$$\begin{aligned}
\mathrm{P}_2 = {} & 1\tfrac{7}{8}\frac{\alpha^2\mathrm{B}}{\mathrm{A}^3} + 3\tfrac{3}{8}\frac{\beta^2}{\mathrm{AB}} + \tfrac{3}{8}\frac{\gamma^2\mathrm{B}}{\mathrm{AC}^2} + 3\tfrac{3}{4}\frac{\alpha\beta}{\mathrm{A}^2} + \tfrac{3}{4}\frac{\alpha\gamma\mathrm{B}}{\mathrm{A}^2\mathrm{C}} + 1\tfrac{1}{4}\frac{\beta\gamma}{\mathrm{AC}} \\
& + \tfrac{3}{8}\frac{\mathrm{L}'\mathrm{B}}{\mathrm{A}^2} + 3\tfrac{3}{8}\frac{\mathrm{M}'}{\mathrm{B}} + \tfrac{3}{8}\frac{\mathrm{N}'\mathrm{B}}{\mathrm{C}^2} + 1\tfrac{1}{4}\frac{l'}{\mathrm{C}} + \tfrac{1}{4}\frac{m'\mathrm{B}}{\mathrm{AC}} + 1\tfrac{1}{4}\frac{n'}{\mathrm{A}} \\
& + \tfrac{1}{4}\frac{\mathrm{BK}_1}{\mathrm{A}} + 1\tfrac{1}{4}\mathrm{K}_2 + \tfrac{1}{4}\frac{\mathrm{BK}_3}{\mathrm{C}} + \tfrac{1}{4}\mathrm{K}_5\mathrm{B} \quad \dots\dots\dots \quad (129)
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_3 = {} & 1\tfrac{7}{8}\frac{\alpha^2\mathrm{C}}{\mathrm{A}^3} + \tfrac{3}{8}\frac{\beta^2\mathrm{C}}{\mathrm{AB}^2} + 3\tfrac{3}{8}\frac{\gamma^2}{\mathrm{AC}} + \tfrac{3}{4}\frac{\alpha\beta\mathrm{C}}{\mathrm{A}^2\mathrm{B}} + 3\tfrac{3}{4}\frac{\alpha\gamma}{\mathrm{A}^2} + 1\tfrac{1}{4}\frac{\beta\gamma}{\mathrm{AB}} \\
& + \tfrac{3}{8}\frac{\mathrm{L}'\mathrm{C}}{\mathrm{A}^2} + \tfrac{3}{8}\frac{\mathrm{M}'\mathrm{C}}{\mathrm{B}^2} + 3\tfrac{3}{8}\frac{\mathrm{N}'}{\mathrm{C}} + 1\tfrac{1}{4}\frac{l'}{\mathrm{B}} + 1\tfrac{1}{4}\frac{m'}{\mathrm{A}} + \tfrac{1}{4}\frac{n'\mathrm{C}}{\mathrm{AB}} \\
& + \tfrac{1}{4}\frac{\mathrm{CK}_1}{\mathrm{A}} + \tfrac{1}{4}\frac{\mathrm{CK}_2}{\mathrm{B}} + 1\tfrac{1}{4}\mathrm{K}_3 + \tfrac{1}{4}\mathrm{K}_5\mathrm{C} \quad \dots\dots\dots \quad (130)
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_4 = {} & \kappa\left(\frac{3\alpha}{\mathrm{A}^2} + \frac{\beta}{\mathrm{AB}} + \frac{\gamma}{\mathrm{AC}}\right) + \frac{p'}{\mathrm{A}} + \frac{q'}{\mathrm{B}} + \frac{r'}{\mathrm{C}} + \mathrm{K}_4 \\
& - \tfrac{1}{4}\left(\tfrac{15}{2}\frac{\alpha^2}{\mathrm{A}^3} + \tfrac{3}{2}\frac{\beta^2}{\mathrm{AB}^2} + \tfrac{3}{2}\frac{\gamma^2}{\mathrm{AC}^2} + \frac{3\alpha\beta}{\mathrm{A}^2\mathrm{B}} + \frac{3\alpha\gamma}{\mathrm{A}^2\mathrm{C}} + \frac{\beta\gamma}{\mathrm{ABC}}\right) \\
& - \tfrac{1}{4}\left(\tfrac{3}{2}\Sigma\frac{\mathrm{L}'}{\mathrm{A}^2} + \Sigma\frac{l'}{\mathrm{BC}} + \frac{\mathrm{K}_1}{\mathrm{A}} + \frac{\mathrm{K}_2}{\mathrm{B}} + \frac{\mathrm{K}_3}{\mathrm{C}} + \mathrm{K}_5\right) \quad \dots\dots \quad (131)
\end{aligned}$$

If the value of ϕ is taken to be $e\phi_1 + e^2\phi_2$, we have as the value of ϕ_2,

$$\begin{aligned}
4\phi_2 = {} & 4u_2 + 4f(w + fw') \\
= {} & \frac{15\alpha^2\xi^4}{\mathrm{A}} + \beta^2\left(\frac{6\xi^2\eta^2}{\mathrm{B}} + \frac{\eta^4}{\mathrm{A}}\right) + \gamma^2\left(\frac{6\xi^2\zeta^2}{\mathrm{C}} + \frac{\zeta^4}{\mathrm{A}}\right) \\
& + \alpha\beta\left(\frac{2\xi^4}{\mathrm{B}} + \frac{12\xi^2\eta^2}{\mathrm{A}}\right) + \alpha\gamma\left(\frac{2\xi^4}{\mathrm{C}} + \frac{12\xi^2\zeta^2}{\mathrm{A}}\right) + \beta\gamma\left(\frac{2\eta^2\zeta^2}{\mathrm{A}} + \frac{2\zeta^2\xi^2}{\mathrm{B}} + \frac{2\xi^2\eta^2}{\mathrm{C}}\right) \\
& + 2\kappa\xi^2\left(\frac{3\alpha}{\mathrm{A}} + \frac{\beta}{\mathrm{B}} + \frac{\gamma}{\mathrm{C}}\right) + 2\kappa\left(\frac{3\alpha\xi^2}{\mathrm{A}} + \frac{\beta\eta^2}{\mathrm{A}} + \frac{\gamma\zeta^2}{\mathrm{A}}\right) + \frac{\kappa^2}{\mathrm{A}} \\
& + \mathrm{L}'\xi^4 + \mathrm{M}'\eta^4 + \mathrm{N}'\zeta^4 + 2l'\eta^2\zeta^2 + 2m'\zeta^2\xi^2 + 2n'\xi^2\eta^2 + 2(p'\xi^2 + q'\eta^2 + r'\zeta^2) + s' \\
& + (\mathrm{A}\xi^2 + \mathrm{B}\eta^2 + \mathrm{C}\zeta^2 - 1)(\mathrm{P}_1\xi^2 + \mathrm{P}_2\eta^2 + \mathrm{P}_3\zeta^2 + \mathrm{P}_4), \quad \dots\dots\dots \quad (132)
\end{aligned}$$

and we have already supposed (§ 25) that

$$\int_0^\infty \frac{\phi_2\, d\lambda}{\Delta} = c_{11}x^4 + c_{22}y^4 + c_{33}z^4 + c_{12}x^2y^2 + c_{13}x^2z^2 + c_{23}y^2z^2 + d_1x^2 + d_2y^2 + d_3z^2 + d_4. \quad (133)$$

Hence, by comparison, we obtain as the values of the coefficients

$$c_{11} = \tfrac{1}{4}\int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{15\alpha^2}{A^5} + \frac{2\alpha\beta}{A^4B} + \frac{2\alpha\gamma}{A^4C} + \frac{L'}{A^4} + \frac{P_1}{A^3}\right) \quad \ldots \ldots \ldots \quad (134)$$

$$c_{22} = \tfrac{1}{4}\int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{\beta^2}{AB^4} + \frac{M'}{B^4} + \frac{P_2}{B^3}\right) \quad \ldots \ldots \ldots \quad (135)$$

$$c_{33} = \tfrac{1}{4}\int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{\gamma^2}{AC^4} + \frac{N'}{C^4} + \frac{P_3}{C^3}\right) \quad \ldots \ldots \ldots \quad (136)$$

$$c_{12} = \tfrac{1}{4}\int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{12\alpha\beta}{A^3B^2} + \frac{6\beta^2}{A^2B^3} + \frac{2\beta\gamma}{A^2B^2C} + \frac{2n'}{A^2B^2} + \frac{P_1}{A^2B} + \frac{P_2}{AB^2}\right) \quad \ldots \quad (137)$$

$$c_{13} = \tfrac{1}{4}\int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{12\alpha\gamma}{A^3C^2} + \frac{6\gamma^2}{A^2C^3} + \frac{2\beta\gamma}{A^2BC^2} + \frac{2m'}{A^2C^2} + \frac{P_1}{A^2C} + \frac{P_3}{AC^2}\right) \quad \ldots \quad (138)$$

$$c_{23} = \tfrac{1}{4}\int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{2\beta\gamma}{AB^2C^2} + \frac{2l'}{B^2C^2} + \frac{P_3}{BC^2} + \frac{P_2}{B^2C}\right) \quad \ldots \ldots \ldots \quad (139)$$

$$d_1 = \tfrac{1}{4}\int_0^\infty \frac{d\lambda}{\Delta}\left\{\kappa\left(\frac{12\alpha}{A^3} + \frac{2\beta}{A^2B} + \frac{2\gamma}{A^2C}\right) + \frac{2p'}{A^2} + \frac{P_4}{A} - \frac{P_1}{A^2}\right\} \quad \ldots \quad (140)$$

$$d_2 = \tfrac{1}{4}\int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{2\kappa\beta}{AB^2} + \frac{2q'}{B^2} + \frac{P_4}{B} - \frac{P_2}{B^2}\right) \quad \ldots \ldots \ldots \quad (141)$$

$$d_3 = \tfrac{1}{4}\int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{2\kappa\gamma}{AC^2} + \frac{2r'}{C^2} + \frac{P_4}{C} - \frac{P_1}{C^2}\right) \quad \ldots \ldots \ldots \quad (142)$$

$$e = \tfrac{1}{4}\int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{\kappa^2}{A} + s' - P_4\right) \quad \ldots \ldots \ldots \quad (143)$$

Evaluation of Certain Integrals.

28. It is clear that before these coefficients can be evaluated, certain integrals must be calculated of the types $J_{BCA\ldots}$, $I_{ABC\ldots}$, where the notation is that of § 21 (equation (67)).

The values of J_A, J_B, J_C have already been evaluated in § 23, as also of I_{AA}, I_{AB}, I_{AC} and I_{AAB}, I_{AAC}, I_{ABC}. We also have

$$I_{BC} = \frac{b^2}{b^2-c^2}J_B - \frac{c^2}{b^2-c^2}J_C = \frac{nb^2}{b^2-c^2} = 0{\cdot}3916228.$$

These 10 integrals will form an adequate basis from which to calculate all the

others. The required integrals can all be obtained by successive applications of formulæ of the following types, which can be verified without trouble:

$$(a^2-b^2)\,J_{AB} = J_B - J_A\,; \qquad (a^2-b^2)\,I_{AB} = I_B - I_A, \text{ \&c.}$$

$$3I_{BB} = 2J_B - I_{BC} - I_{AB}\,; \qquad a^2\,J_{AA} = J_A - I_{AA}, \text{ \&c.}$$

$$(2n+1)\,J_{A^{n+1}} = \frac{2}{abc\,.\,a^{2n}} - J_{A^nB} - J_{A^nC},$$

$$(2n+1)\,I_{A^{n+1}} = 2\,J_{A^n} - I_{A^nB} - I_{A^nC}.$$

A great number of the integrals can be calculated in two or more ways, and owing to this circumstance it is possible to provide very complete checks on the accuracy of calculation. Two complications are worthy of mention.

In the first place one must consider the ordinary cumulative error of all prolonged computation. Since b^2 is nearly equal to c^2, any error present will be increased when we evaluate a new quantity of the type $\frac{f(b^2)-f(c^2)}{b^2-c^2}$; consequently where a quantity can be evaluated in two ways, the one in which division by b^2-c^2 is not involved has been taken to be the true value, and the one derived by division by b^2-c^2, has been used merely as a check, and has generally been found to differ in the sixth or seventh place (or near the end of the computations even in the fifth place) from the other values.

Secondly, if the 10 integrals used as base were known with perfect accuracy, the checks ought to be satisfied fully except for the error in the last one or two figures. But, as has been indicated in § 23, the 10 integrals are not themselves perfectly self-consistent, so that different methods of computation will lead to a difference of the final results comparable with the errors in the basic integrals.

The following table gives the values I have selected as the best for the various integrals required. I have not thought it necessary to record the checks or estimate the probable errors here, as a much more searching test of the accuracy of the whole computation can be provided at a later stage.

$$J = 1{\cdot}8401326.$$

$$J_A = 0{\cdot}2583003, \qquad J_B = 0{\cdot}7647290, \qquad J_C = 0{\cdot}9769708,$$
$$J_{AA} = 0{\cdot}05262769, \qquad J_{AB} = 0{\cdot}1751040, \qquad J_{AC} = 0{\cdot}2293883,$$
$$J_{BB} = 0{\cdot}6516017, \qquad J_{BC} = 0{\cdot}8813026, \qquad J_{CC} = 1{\cdot}2044842,$$

$$J_{AAA} = 0{\cdot}011873224, \qquad J_{AAB} = 0{\cdot}04234772, \qquad J_{AAC} = 0{\cdot}05641920,$$
$$J_{ABB} = 0{\cdot}1647550, \qquad J_{ABC} = 0{\cdot}2254075, \qquad J_{ACC} = 0{\cdot}3112352,$$

$$J_{BBB} = 0{\cdot}6830283, \qquad J_{BBC} = 0{\cdot}9537991, \qquad J_{BCC} = 1{\cdot}9011148, \qquad J_{CCC} = 1{\cdot}9011148,$$

$J_{AAAA} = 0{\cdot}00281568,\quad J_{AAAB} = 0{\cdot}01053694,\quad J_{AAAC} = 0{\cdot}01421838,$
$J_{AABB} = 0{\cdot}04232384,\quad J_{AABC} = 0{\cdot}05842981,\quad J_{AACC} = 0{\cdot}08133328,$
$J_{ABBB} = 0{\cdot}1791995,\quad J_{ABBC} = 0{\cdot}2518508,\quad J_{ABCC} = 0{\cdot}3563870,$
$J_{ACCC} = 0{\cdot}5074646,\quad J_{BBBB} = 0{\cdot}788915,\quad J_{BBBC} = 1{\cdot}124336,$
$J_{BBCC} = 1{\cdot}611794,\quad J_{BCCC} = 2{\cdot}321795,\quad J_{CCCC} = 3{\cdot}361221,$

$J_{AAAAA} = 0{\cdot}0006881964,\quad J_{AAAAB} = 0{\cdot}002669728,\quad J_{AAAAC} = 0{\cdot}003639559,$
$J_{AAABB} = 0{\cdot}01099071,\quad J_{AAABC} = 0{\cdot}01528665,\quad J_{AAACC} = 0{\cdot}02142202.$
$J_{AABBB} = 0{\cdot}04732648,\quad J_{AABBC} = 0{\cdot}06687764,\quad J_{AABCC} = 0{\cdot}09532252.$
$J_{AACCC} = 0{\cdot}1360144,\quad J_{ABBBB} = 0{\cdot}2108169,\quad J_{ABBBC} = 0{\cdot}3016727,$
$J_{ABBCC} = 0{\cdot}4340725,\quad J_{ABCCC} = 0{\cdot}6273273,\quad J_{ACCCC} = 0{\cdot}910874,$

$I_{A} = 0{\cdot}9215282,\quad I_{B} = 1{\cdot}3322118,\quad I_{C} = 1{\cdot}4265252,$
$I_{AA} = 0{\cdot}0711382,\quad I_{AB} = 0{\cdot}1419990,\quad I_{AC} = 0{\cdot}1611871,$
$I_{BB} = 0{\cdot}3319454,\quad I_{BC} = 0{\cdot}3916228,\quad I_{CC} = 0{\cdot}4670439,$

$I_{AAA} = 0{\cdot}01040244,\quad I_{AAB} = 0{\cdot}02450100,\quad I_{AAC} = 0{\cdot}02874219,$
$I_{ABB} = 0{\cdot}06567633,\quad I_{ABC} = 0{\cdot}07967602,\quad I_{ACC} = 0{\cdot}09762467,$

$I_{BBB} = 0{\cdot}19794510,\quad I_{BBC} = 0{\cdot}2478016,\quad I_{BCC} = 0{\cdot}3131750,\quad I_{CCC} = 0{\cdot}3996337,$

$I_{AAAA} = 0{\cdot}001859707,\quad I_{AAAB} = 0{\cdot}004874751,\quad I_{AAAC} = 0{\cdot}005853759,$
$I_{AABB} = 0{\cdot}01423688,\quad I_{AABC} = 0{\cdot}01761100,\quad I_{AACC} = 0{\cdot}02198620,$
$I_{ABBB} = 0{\cdot}04573358,\quad I_{ABBC} = 0{\cdot}05813153,\quad I_{ABCC} = 0{\cdot}07452920,$
$I_{ACCC} = 0{\cdot}0963965,\quad I_{BBBB} = 0{\cdot}1590430,\quad I_{BBBC} = 0{\cdot}2070218,$
$I_{BBCC} = 0{\cdot}2714534,\quad I_{BCCC} = 0{\cdot}3590070,\quad I_{CCCC} = 0{\cdot}4811801,$

Evaluation of the Coefficients $c_{11}, c_{12}, \ldots$

29. It will be noticed that the coefficients c_{11}, c_{12}, ... are linear in α^2, $\alpha\beta$, ..., L′, M′, N′, ..., p', q', r', s' so that the various contributions may be calculated separately and independently.

Contributions from Terms in p', q', r', s'.

As regards the contributions from these coefficients, we may take (*cf.* equations (125) and (126))

$$K_1 = K_2 = K_3 = 0\,;\qquad K_4 = -\Sigma\frac{p'}{a^2}\,;\qquad K_5 = 0,$$

whence we obtain as the contributions to P_1, P_2, P_3, P_4 (*cf.* equations (128) to (131),

$$P_1 = P_2 = P_3 = 0,$$

$$P_4 = \frac{p'}{A}+\frac{q'}{B}+\frac{r'}{C}+K_4 = -\lambda\left(\frac{p'}{a^2A}+\frac{q'}{b^2B}+\frac{r'}{c^2C}\right).$$

It now appears that p', q', r', s' contribute nothing to the values of c_{11}, c_{12}, $\dots$, c_{23} (*cf.* equations (134) to (139), p. 60.) Their contributions to $4d_1$, $4d_2$, and $4d_3$ are as follows:

$$4d_1 = \int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{2p'}{A^2}+\frac{P_4}{A}\right)$$

$$= \int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{2p'}{A^2}-\frac{p'}{a^2}\frac{\lambda}{A^2}-\frac{q'}{b^2}\frac{\lambda}{AB}-\frac{r'}{c^2}\frac{\lambda}{AC}\right)$$

so that the contributions are

$$4d_1 = 2p'\,J_{AA}-\frac{p'}{a^2}I_{AA}-\frac{q'}{b^2}I_{AB}-\frac{r'}{c^2}I_{AC},$$

$$4d_2 = 2q'\,J_{BB}-\frac{p'}{a^2}I_{AB}-\frac{q'}{b^2}I_{BB}-\frac{r'}{c^2}I_{BC},$$

$$4d_3 = 2r'\,J_{CC}-\frac{p'}{a^2}I_{AC}-\frac{q'}{b^2}I_{BC}-\frac{r'}{c^2}I_{CC}.$$

Since this part of the potential should be harmonic, we ought to have $d_1+d_2+d_3 = 0$ (*cf.* equation (111)). This is clearly the case, in virtue of the identity

$$2a^2\,J_{AA} = I_{AA}+I_{AB}+I_{AC}.$$

I have verified that these identities are satisfied by the values in the table opposite, and the contributions are found to be

$$4d_1 = \quad 0{\cdot}303186\frac{p'}{a^2}-0{\cdot}141999\frac{q'}{b^2}-0{\cdot}161187\frac{r'}{c^2}$$

$$4d_2 = -0{\cdot}141999\frac{p'}{a^2}+0{\cdot}533622\frac{q'}{b^2}-0{\cdot}391623\frac{r'}{c^2},$$

$$4d_3 = -0{\cdot}161187\frac{p'}{a^2}-0{\cdot}391623\frac{q'}{b^2}+0{\cdot}552810\frac{r'}{c^2}.$$

Contributions from Terms in L', M', N_1, l', m', n'.

30. As regards these terms, we may take (*cf.* equations (125) and (126)),

$$K_1 = -\left(\frac{3L'}{a^2}+\frac{n'}{b^2}+\frac{m'}{c^2}\right)$$

$$K_2 = -\left(\frac{n'}{a^2}+\frac{3M'}{b^2}+\frac{l'}{c^2}\right)$$

$$K_3 = -\left(\frac{m'}{a^2}+\frac{l'}{b^2}+\frac{3N'}{c^2}\right)$$

$$K_4 = 0,$$

$$K_5 = -\tfrac{3}{2}\Sigma\frac{L'}{a^4}-\Sigma\frac{l'}{b^2c^2}-\Sigma\frac{K_1}{a^2} = \tfrac{3}{2}\Sigma\frac{L'}{a^4}+\Sigma\frac{l'}{b^2c^2}$$

and the contributions to P_1, P_2, P_3, P_4, are found to be (*cf.* equations (128) to (131)),

$$P_4 = -\tfrac{1}{4}\left(\tfrac{3}{2}\Sigma\frac{L'}{a^4}\frac{\lambda^2}{A^2}+\Sigma\frac{l'}{b^2c^2}\frac{\lambda^2}{BC}\right)$$

$$P_1 = -\left(\frac{3L'\lambda}{a^2A}+\frac{m'\lambda}{c^2C}+\frac{n'\lambda}{b^2B}\right)-AP_4$$

$$P_2 = -\left(\frac{l'\lambda}{c^2C}+\frac{3M'\lambda}{b^2B}+\frac{n'\lambda}{a^2A}\right)-BP_4$$

$$P_3 = -\left(\frac{l'\lambda}{b^2B}+\frac{m'\lambda}{a^2A}+\frac{3N'\lambda}{c^2C}\right)-CP_4.$$

We find as the contribution to d_1, (equation (140), p. 60)

$$4d_1 = \int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{P_4}{A}-\frac{P_1}{A^2}\right) = \int_0^\infty \frac{d\lambda}{\Delta}\left\{\frac{2P_4}{A}+\frac{1}{A^2}\left(\frac{3L'\lambda}{a^2A}+\frac{m'\lambda}{c^2C}+\frac{n'\lambda}{b^2B}\right)\right\}, \text{ \&c.,}$$

from which it is easily verified that $d_1+d_2+d_3 = 0$, as it ought to be.

In virtue of this relation it is only necessary to evaluate two of the contributions $4d_1$, $4d_2$, $4d_3$, but I have calculated all three directly from the table on p. 61, so as to obtain a check on the amount of error involved from the causes mentioned in § 28, as well as a check on the accuracy of my own computations. The values I find are

$$\begin{aligned}4d_1 = {} & 0{\cdot}85375\frac{L'}{a^4} - 0{\cdot}073783\frac{M'}{b^4} - 0{\cdot}089893\frac{N'}{c^4} \\ & - 0{\cdot}054134\frac{l'}{b^2c^2} + 0{\cdot}072732\frac{m'}{c^2a^2} + 0{\cdot}059701\frac{n'}{a^2b^2},\end{aligned}$$

$$\begin{aligned}4d_2 = {} & -0{\cdot}041149\frac{L'}{a^4} + 0{\cdot}244072\frac{M'}{b^4} - 0{\cdot}194266\frac{N'}{c^4} \\ & + 0{\cdot}051969\frac{l'}{b^2c^2} - 0{\cdot}054134\frac{m'}{c^2a^2} - 0{\cdot}005568\frac{n'}{a^2b^2},\end{aligned}$$

$$\begin{aligned}4d_3 = {} & -0{\cdot}044227\frac{L'}{a^4} - 0{\cdot}170277\frac{M'}{b^4} + 0{\cdot}284157\frac{N'}{c^4} \\ & + 0{\cdot}003091\frac{l'}{b^2c^2} - 0{\cdot}018600\frac{m'}{c^2a^2} - 0{\cdot}054134\frac{n'}{a^2b^2}.\end{aligned}$$

From these figures the value of $d_1+d_2+d_3$ would be given by

$$\begin{aligned}4(d_1+d_2+d_3) = {} & -0{\cdot}000001\frac{L'}{a^4} + 0{\cdot}000012\frac{M'}{b^4} - 0{\cdot}000002\frac{N'}{c^4} \\ & + 0{\cdot}000026\frac{l'}{b^2c^2} - 0{\cdot}000002\frac{m'}{a^2c^2} - 0{\cdot}000001\frac{n'}{a^2b^2}.\end{aligned}$$

This check gives an idea of the amount of error involved. It illustrates the tendency of the errors to accumulate in terms where b's and c's are plentiful, and to be absent

where a's are present. This is a consequence of the method of procedure explained in § 28.

In calculating the contributions to c_{11}, c_{12}, ..., I have (unwisely) taken the variables to be L′, M′, ..., instead of $\frac{L'}{a^4}$, $\frac{M'}{b^4}$, ... In virtue of relations (108) to (110), only three contributions need have been calculated, but I have calculated five out of the six quite independently, so as to have two independent checks on the computations.

For the contribution to c_{11}, I find

$$
\begin{aligned}
4c_{11} &= \int_0^\infty \frac{d\lambda}{\Delta}\left(\frac{L'}{A^4}+\frac{P_1}{A^3}\right)\\
&= L'\left(J_{AAAA}-\frac{27}{8a^2}I_{AAAA}+\frac{3}{8a^4}I_{AAA}\right)+M'\left(\frac{3}{8b^4}I_{AAB}-\frac{3}{8b^2}I_{AABB}\right)\\
&\quad+N'\left(\frac{3}{8c^4}I_{AAC}-\frac{3}{8c^2}I_{AACC}\right)+l'\left(\frac{1}{4b^2c^2}I_{AAC}-\frac{1}{4c^2}I_{AABC}\right)\\
&\quad+m'\left(\frac{1}{4c^2a^2}I_{AAA}-\frac{1}{4a^2}I_{AAAC}-\frac{1}{c^2}I_{AAAC}\right)+n'\left(\frac{1}{4a^2b^2}I_{AAB}-\frac{5}{4b^2}I_{AAAB}\right).
\end{aligned}
$$

The other five contributions can be similarly written down. It then can be verified algebraically that the three relations (108) to (110) are satisfied; this provides a check on the method and the algebra, but it hardly seems worth exhibiting it here, as the numerical checks to be given later provide simultaneously checks on the method, the algebra, and the computations. By independent computations I find for the contributions to c_{11}, c_{22}, c_{33}, c_{12}, c_{13}, the following:

$$
\begin{aligned}
4c_{11} &= 0{\cdot}001359233L'+0{\cdot}01278935M'+0{\cdot}04066156N'\\
&\qquad+0{\cdot}01515472l'-0{\cdot}01251121m'-0{\cdot}006581149n'\\
4c_{22} &= 0{\cdot}000446082L'+0{\cdot}1490178M'+0{\cdot}2780161N'\\
&\qquad-0{\cdot}3909331l'+0{\cdot}00681885m'-0{\cdot}00912350n'\\
4c_{33} &= 0{\cdot}000576222L'+0{\cdot}1129554M'+0{\cdot}385800N'\\
&\qquad-0{\cdot}3972137l'-0{\cdot}01767165m'+0{\cdot}00509336n'\\
4c_{12} &= -0{\cdot}003687295L'-0{\cdot}1465540M'+0{\cdot}2013742N'\\
&\qquad-0{\cdot}0642884l'-0{\cdot}03593784m'+0{\cdot}06239405n'\\
4c_{13} &= -0{\cdot}004468110L'+0{\cdot}0698178M'-0{\cdot}4453435N'\\
&\qquad-0{\cdot}02663960l'+0{\cdot}11100470m'-0{\cdot}02290710n'.
\end{aligned}
$$

From these

$$
\begin{aligned}
4(6c_{11}+c_{12}+c_{13}) &= -0{\cdot}000000007L'-0{\cdot}0000001M'+0{\cdot}0000000N'\\
&\qquad-0{\cdot}0000003l'-0{\cdot}0000004m'+0{\cdot}00000000n',
\end{aligned}
$$

I have not calculated c_{23} independently, but a check is provided by the comparison of the values of c_{23} deduced from the above figures by the use of equations (109) and (110)

separately. For these values I find the following, the upper value being that derived from equation (109):

$$4c_{23} = \frac{0·00101080}{0·00101078}\mathrm{L}' - \frac{0·747552}{0·747550}\mathrm{M}' - \frac{1·86947}{1·86946}\mathrm{N}'$$

$$+\frac{2·40989}{2·40992}l' - \frac{0·0049753}{0·0049754}m' - \frac{0·0076531}{0·0076515}n'.$$

Contribution from terms in α^2, $\alpha\beta$, ..., &c.

31. For the computation of these contributions it is convenient to take the standard set of value obtained in § 24, namely

$$\alpha' = \frac{\alpha}{a^2} = -1, \qquad \beta' = \frac{\beta}{b^2} = 0·3081810, \qquad \gamma' = \frac{\gamma}{c^2} = 0·1604294,$$

$$\kappa = -\tfrac{1}{5}(3\alpha'+\beta'+\gamma') = 0·506278.$$

These give the values

$$\alpha = -3·556343, \qquad \beta = 0·2046890, \qquad \gamma = 0·06791892,$$

$$\alpha^2 = 12·647573, \qquad \beta^2 = 0·0418976, \qquad \gamma^2 = 0·004612981,$$

$$\alpha\beta = -0·7279442, \qquad \alpha\gamma = -0·2415430, \qquad \beta\gamma = 0·01390225.$$

Then, as regards the terms in α^2, $\alpha\beta$, ..., &c., including κ, we find (*cf.* equations (125) and (126))

$$\mathrm{K}_1 = -\left(\tfrac{45}{2}\frac{\alpha^2}{a^4} + \tfrac{3}{2}\frac{\beta^2}{b^4} + \tfrac{3}{2}\frac{\gamma^2}{c^4} + \frac{6\alpha\beta}{a^2b^2} + \frac{6\alpha\gamma}{a^2c^2} + \frac{\beta\gamma}{b^2c^2}\right)$$

$$\mathrm{K}_2 = -\left(\frac{3\beta^2}{a^2b^2} + \frac{3\alpha\beta}{a^4} + \frac{\beta\gamma}{a^2c^2}\right) = -\frac{b^2}{a^2}\beta'(3\alpha'+3\beta'+\gamma')$$

$$\mathrm{K}_3 = -\left(\frac{3\gamma^2}{a^2c^2} + \frac{3\alpha\gamma}{a^4} + \frac{\beta\gamma}{a^2b^2}\right) = -\frac{c^2}{a^2}\gamma'(3\alpha'+\beta'+3\gamma')$$

$$\mathrm{K}_4 = -\frac{\kappa}{a^2}\left(\frac{3\alpha}{a^2} + \frac{\beta}{b^2} + \frac{\gamma}{c^2}\right) = \frac{5\kappa^2}{a^2}$$

$$\mathrm{K}_5 = -\left(\tfrac{15}{2}\frac{\alpha^2}{a^6} + \tfrac{3}{2}\frac{\beta^2}{a^2b^4} + \tfrac{3}{2}\frac{\gamma^2}{a^2c^4} + \frac{3\alpha\beta}{a^4b^2} + \frac{3\alpha\gamma}{a^4c^2} + \frac{\beta\gamma}{a^2b^2c^2} + \frac{\mathrm{K}_1}{a^2} + \frac{\mathrm{K}_2}{b^2} + \frac{\mathrm{K}_3}{c^2}\right)$$

$$= -\tfrac{2}{3}\left(\frac{\mathrm{K}_1}{a^2} + \frac{\mathrm{K}_2}{b^2} + \frac{\mathrm{K}_3}{c^2}\right),$$

giving on substitution of numerical values, as regards terms in α^2, $\alpha\beta$, ... only

$$\mathrm{K}_1 = -19·91885, \qquad \mathrm{K}_2 = 0·1102214, \qquad \mathrm{K}_3 = 0·04221664,$$

$$\mathrm{K}_4 = 0·3603665, \qquad \mathrm{K}_5 = 3·556845.$$

The values of P_1, P_2, P_3, P_4 originating from these terms are

$$\frac{P_1}{A^3} = \frac{1}{A^3}\left\{24\tfrac{3}{8}\frac{\alpha^2}{A^2} + 1\tfrac{7}{8}\frac{\beta^2}{B^2} + 1\tfrac{7}{8}\frac{\gamma^2}{C^2} + 6\tfrac{3}{4}\frac{\alpha\beta}{AB} + 6\tfrac{3}{4}\frac{\alpha\gamma}{AC} + 1\tfrac{1}{4}\frac{\beta\gamma}{BC}\right\} + \frac{1}{A^2}\left\{1\tfrac{1}{4}\frac{K_1}{A} + \tfrac{1}{4}\frac{K_2}{B} + \tfrac{1}{4}\frac{K_3}{C} + \tfrac{1}{4}K_5\right\}$$

$$\frac{P_2}{B^3} = \frac{1}{AB^2}\left\{1\tfrac{7}{8}\frac{\alpha^2}{A^2} + 3\tfrac{3}{8}\frac{\beta^2}{B^2} + \tfrac{3}{8}\frac{\gamma^2}{C^2} + 3\tfrac{3}{4}\frac{\alpha\beta}{AB} + \tfrac{3}{4}\frac{\alpha\gamma}{AC} + 1\tfrac{1}{4}\frac{\beta\gamma}{BC}\right\} + \frac{1}{B^2}\left\{\tfrac{1}{4}\frac{K_1}{A} + 1\tfrac{1}{4}\frac{K_2}{B} + \tfrac{1}{4}\frac{K_3}{C} + \tfrac{1}{4}K_5\right\}$$

$$\frac{P_3}{C^3} = \frac{1}{AC^2}\left\{1\tfrac{7}{8}\frac{\alpha^2}{A^2} + \tfrac{3}{8}\frac{\beta^2}{B^2} + 3\tfrac{3}{8}\frac{\gamma^2}{C^2} + \tfrac{3}{4}\frac{\alpha\beta}{AB} + 3\tfrac{3}{4}\frac{\alpha\gamma}{AC} + 1\tfrac{1}{4}\frac{\beta\gamma}{BC}\right\} + \frac{1}{C^2}\left\{\tfrac{1}{4}\frac{K_1}{A} + \tfrac{1}{4}\frac{K_2}{B} + 1\tfrac{1}{4}\frac{K_3}{C} + \tfrac{1}{4}K_5\right\}$$

$$P_4 = \kappa\left(\frac{3\alpha}{A^2} + \frac{\beta}{AB} + \frac{\gamma}{AC}\right) + K_4 - \tfrac{1}{4}\left(\tfrac{15}{2}\frac{\alpha^2}{A^3} + \frac{3\beta^2}{2AB^2} + \frac{3\gamma^2}{2AC^2} + \frac{3\alpha\beta}{A^2B} + \frac{3\alpha\gamma}{A^2C} + \frac{\beta\gamma}{ABC}\right) - \tfrac{1}{4}\left(\frac{K_1}{A} + \frac{K_2}{B} + \frac{K_3}{C} + K_5\right).$$

I have evaluated independently the three contributions to d_1, d_2, d_3 and find for them

$$\begin{aligned}4d_1 = {} & 15\alpha\kappa J_{AAA} + 3\beta\kappa J_{AAB} + 3\gamma\kappa J_{AAC} + K_4 J_A \\ & -(26\tfrac{1}{4}\alpha^2 J_{AAAA} + 2\tfrac{1}{4}\beta^2 J_{AABB} + 2\tfrac{1}{4}\gamma^2 J_{AACC} + 7\tfrac{1}{2}\alpha\beta J_{AAAB} + 7\tfrac{1}{2}\alpha\gamma J_{AAAC} + 1\tfrac{1}{2}\beta\gamma J_{AABC}) \\ & -(1\tfrac{1}{2}K_1 J_{AA} + \tfrac{1}{2}K_2 J_{AB} + \tfrac{1}{2}K_3 J_{AC} + \tfrac{1}{2}K_5 J_A),\end{aligned}$$

and there are corresponding values for the contributions to $4d_2$ and $4d_3$. The numerical values found by independent computations are

$$4d_1 = 0{\cdot}032398, \qquad 4d_2 = -0{\cdot}011175, \qquad 4d_3 = -0{\cdot}021223.$$

As a check on the computations, the sum of these contributions ought to be zero; we find

$$4\,(d_1+d_2+d_3) = 0{\cdot}000000.$$

The six contributions to c_{11}, c_{12}, ... have also all been calculated separately. The values obtained are

$$4c_{11} = 0{\cdot}0722711, \qquad 4c_{22} = 0{\cdot}0271162, \qquad 4c_{33} = 0{\cdot}0274693,$$
$$4c_{12} = -0{\cdot}215756, \qquad 4c_{13} = -0{\cdot}217869, \qquad 4c_{23} = 0{\cdot}053056,$$

as regards the checks which ought to be satisfied by these values, I find

$$6c_{11}+c_{12}+c_{13} = 0{\cdot}000002, \quad 6c_{22}+c_{12}+c_{23} = -0{\cdot}000003, \quad 6c_{33}+c_{13}+c_{23} = +0{\cdot}000003.$$

Formation and Solution of the Equation.

32. We proceed to form and solve the final equations. It is convenient to deal with the equations in c_{11}, c_{12} ... first.

By comparison of coefficients in equation (123), we obtain

$$L = L' + \frac{9\alpha^2}{a^2} - 10\,\kappa\alpha \qquad = L' + 50{\cdot}01206 \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

$$M = M' + \frac{\beta^2}{a^2} \qquad = M' + 0{\cdot}01178109 \quad . \quad . \quad . \quad . \quad (145)$$

$$N = N' + \frac{\gamma^2}{a^2} \qquad = N' + 0{\cdot}001297113 \quad . \quad . \quad . \quad . \quad (146)$$

$$l = l' + \frac{\beta\gamma}{a^2} \qquad = l' + 0{\cdot}003909145 \quad . \quad . \quad . \quad . \quad (147)$$

$$m = m' + \frac{3\alpha\gamma}{a^2} + \frac{2\gamma^2}{c^2} - 5\kappa\gamma \qquad = m' - 0{\cdot}3538936 \quad . \quad . \quad . \quad . \quad . \quad (148)$$

$$n = n' + \frac{3\alpha\beta}{a^2} + \frac{2\beta^2}{b^2} - 5\kappa\beta \qquad = n' - 1{\cdot}0060520 \quad . \quad . \quad . \quad . \quad . \quad (149)$$

$$p = p' + \frac{3\alpha\kappa}{a^2} - 5\kappa^2 \qquad = p' - 2{\cdot}800420 \quad . \quad . \quad . \quad . \quad . \quad (150)$$

$$q = q' + \frac{\beta\kappa}{a^2} \qquad = q' + 0{\cdot}02913935 \quad . \quad . \quad . \quad . \quad (151)$$

$$r = r' + \frac{\gamma\kappa}{a^2} \qquad = r' + 0{\cdot}009668880 \quad . \quad . \quad . \quad . \quad (152)$$

$$s = s' + \frac{\kappa^2}{a^2} \qquad = s' + 0{\cdot}07207330 \quad . \quad . \quad . \quad . \quad (153)$$

These coefficients can be checked by comparing the sums of the coefficients in the two values of ω' (equation (123)); *i.e.*, taking $\xi^2 = \eta^2 = \zeta^2 = 1$. For the difference of the sums which ought to vanish, I find $41{\cdot}86192 - 41{\cdot}86191 = 0{\cdot}00001$. Using the values just obtained,

$$\frac{\theta}{a^8}\,L = 0{\cdot}002585675L' + 0{\cdot}1293149,$$

$$\frac{\theta}{b^8}\,M = 2{\cdot}125367M' + 0{\cdot}02503914,$$

$$\frac{\theta}{c^8}\,N = 12{\cdot}87541N' + 0{\cdot}01670087.$$

Collecting the various contributions to c_{11}, c_{22}, c_{33}, equations (102) to (104) now become

$$\begin{aligned} 0{\cdot}002585675L' + 0{\cdot}1293149 \\ = 0{\cdot}001359233L' + 0{\cdot}01278935M' + 0{\cdot}04066156N' \\ + 0{\cdot}01515472l' - 0{\cdot}01251121m' - 0{\cdot}006581149n' + 0{\cdot}0722711, \end{aligned}$$

$$2{\cdot}125367\mathrm{M}'+0{\cdot}02503914$$
$$=0{\cdot}000446082\mathrm{L}'+0{\cdot}1490178\mathrm{M}'+0{\cdot}2780161\mathrm{N}'$$
$$-0{\cdot}3909331l'+0{\cdot}00681885m'-0{\cdot}00912350n'+0{\cdot}0271162,$$

$$12{\cdot}87541\mathrm{N}'+0{\cdot}01670087$$
$$=0{\cdot}000576222\mathrm{L}'+0{\cdot}1129554\mathrm{M}'+0{\cdot}385800\mathrm{N}'$$
$$-0{\cdot}3972137l'-0{\cdot}01767165m'+0{\cdot}00509336n'+0{\cdot}0274693,$$

while equations (115) to (117) give

$$\mathrm{L}'=-23{\cdot}52190m'-9{\cdot}55672n'-32{\cdot}07326,$$
$$\mathrm{M}'=-\ 0{\cdot}8204313l'-0{\cdot}01162649n'-0{\cdot}00329142,$$
$$\mathrm{N}'=-\ 0{\cdot}1354301l'-0{\cdot}00472373m'-0{\cdot}000154830.$$

The solution of this set of six equation is

$$\mathrm{L}'=-61{\cdot}72711,\qquad \mathrm{M}'=-0{\cdot}01761613,\qquad \mathrm{N}'=-0{\cdot}002105706,$$
$$l'=-\ 0{\cdot}006053622,\qquad m'=+0{\cdot}5865530,\qquad n'=+1{\cdot}659250. \quad (154)$$

No check is needed on the accuracy of these values beyond the fact that they satisfy the equations. On substituting directly into the equations, I find that they are all satisfied accurately to the last place of decimals.

The corresponding values of L, M, N, l, m, n (see opposite page) are

$$\mathrm{L}=-11{\cdot}71505.\qquad \mathrm{M}=-0{\cdot}00583504,\qquad \mathrm{N}=-0{\cdot}000808592,$$
$$l=-\ 0{\cdot}002144477,\qquad m=\ 0{\cdot}2326594,\qquad n=\ 0{\cdot}653198, \quad (155)$$

and the values of c_{11}, c_{12}, ..., calculated directly from equations (102) to (107) are

$$4c_{11}=-0{\cdot}03029132,\qquad 4c_{22}=-0{\cdot}01240160,\qquad 4c_{33}=-0{\cdot}010410956,$$
$$2c_{23}=-0{\cdot}01121809,\qquad 2c_{31}=\ 0{\cdot}04245102,\qquad 2c_{12}=\ 0{\cdot}04842301\,. \quad (156)$$

These ought to satisfy the checks afforded by equations (108) to (110). In point of fact, I find

$$6c_{11}+c_{12}+c_{13}=+0{\cdot}00000007,\qquad 6c_{22}+c_{23}+c_{12}=+0{\cdot}00000012,$$
$$6c_{33}+c_{13}+c_{23}=+0{\cdot}00000006.$$

33. The first use which must be made of these numbers is to determine the contributions to d_1, d_2, d_3, evaluated on p. 64. We have by separate computation,

$$0{\cdot}085375\,\frac{\mathrm{L}'}{a^4}-0{\cdot}073783\,\frac{\mathrm{M}'}{b^4}-0{\cdot}089893\,\frac{\mathrm{N}'}{c^4}-\ldots\quad=-0{\cdot}3412367,$$

$$-0{\cdot}041149\,\frac{\mathrm{L}'}{a^4}+\ldots\quad=+0{\cdot}1672652,$$

$$-0{\cdot}044227\,\frac{\mathrm{L}'}{a^4}+\ldots\quad=+0{\cdot}1739738,$$

Hence collecting all contributions to $4d_1$, $4d_2$, $4d_3$, we obtain as their total values

$$4d_1 = \quad 0{\cdot}303186\frac{p'}{a^2} - 0{\cdot}141999\frac{q'}{b^2} - 0{\cdot}161187\frac{r'}{c^2} - 0{\cdot}308839,$$

$$4d_2 = -0{\cdot}141999\frac{p'}{a^2} + 0{\cdot}533622\frac{q'}{b^2} - 0{\cdot}391623\frac{r'}{c^2} + 0{\cdot}156090,$$

$$4d_3 = -0{\cdot}161187\frac{p'}{a^2} - 0{\cdot}391623\frac{q'}{b^2} + 0{\cdot}552810\frac{r'}{c^2} + 0{\cdot}152751.$$

From the values already obtained for p, q, r, we can transform equations (112) to (114) into the following :—

$$-4n'' = \theta\left(\frac{p}{a^4}+\frac{q}{b^4}+\frac{r}{c^4}\right) = 0{\cdot}1163013\frac{p'}{a^2} + 0{\cdot}6227300\frac{q'}{b^2} + 0{\cdot}9769708\frac{r'}{c^2} - 0{\cdot}0419475,$$

$$4\left(d_1 - d_2\right) = 2\theta\left(\frac{p}{a^4}-\frac{q}{b^4}\right) = 0{\cdot}2326027\frac{p'}{a^2} - 1{\cdot}2454600\frac{q'}{b^2} - 0{\cdot}2378029,$$

$$4d_3 = 2\theta\frac{r}{c^4} = 1{\cdot}9539416\frac{r'}{c^2} + 0{\cdot}04462528.$$

On substituting the values of $4d_1$, $4d_2$, and $4d_3$, these reduce to

$$0{\cdot}212582\frac{p'}{a^2} + 0{\cdot}569839\frac{q'}{b^2} + 0{\cdot}230436\frac{r'}{c^2} = 0{\cdot}227126,$$

$$0{\cdot}161187\frac{p'}{a^2} + 0{\cdot}391623\frac{q'}{b^2} + 1{\cdot}401132\frac{r'}{c^2} = 0{\cdot}108126,$$

$$0{\cdot}116301\frac{p'}{a^2} + 0{\cdot}622730\frac{q'}{b^2} + 0{\cdot}976971\frac{r'}{c^2} = 0{\cdot}0419475 - 4n''.$$

The solution of these equations is found to be

$$\frac{p'}{a^2} = 1{\cdot}428257 + 32{\cdot}04689n'' \quad \ldots \quad (157)$$

$$\frac{q'}{b^2} = -0{\cdot}111620 - 11{\cdot}797935n'' \quad \ldots \quad (158)$$

$$\frac{r'}{c^2} = -0{\cdot}0559390 - 0{\cdot}3891163n'' \quad \ldots \quad (159)$$

and this solution has been verified by direct insertion into all the equations.

34. This completes the solution of all the equations, and the determination of all the coefficients except s'.

On collecting the values of K_1, K_2, K_3, K_4, K_5, from pp. 62, 63, and 66, and inserting the numerical values already obtained, we find

$$K_1 = -\left(\frac{3L'}{a^2}+\frac{n'}{b^2}+\frac{m'}{c^2}\right)-19{\cdot}91885 = 28{\cdot}26823,$$

$$K_2 = -\left(\frac{n'}{a^2}+\frac{3M'}{b^2}+\frac{l'}{c^2}\right)+0{\cdot}1102214 = -0{\cdot}2624723,$$

$$K_3 = -\left(\frac{m'}{a^2}+\frac{l'}{b^2}+\frac{3N'}{c^2}\right)+0{\cdot}04221664 = -0{\cdot}0986790,$$

$$K_4 = -\frac{p'}{a^2}-\frac{q'}{b^2}-\frac{r'}{c^2}+0{\cdot}3603665 = -0{\cdot}900332-19{\cdot}85984n'',$$

$$K_5 = \tfrac{3}{2}\Sigma\frac{L'}{a^4}+\Sigma\frac{l'}{b^2c^2}+3{\cdot}556845 = -2{\cdot}770998.$$

To evaluate s' we have to examine the form assumed by the potential at infinity. The additional terms in the potential, as far as terms in $\frac{1}{r}$, produced by the distortion are readily found to be

$$-\pi\rho abc\frac{2}{r}(s'-\tfrac{2}{3}K_4+\tfrac{2}{15}K_5),$$

and if δm is the additional mass produced by the distortion, this must be identical with $\frac{\delta m}{r}$. Hence, if s' is determined by the condition of constancy of mass, we have

$$s' = \tfrac{2}{3}K_4-\tfrac{2}{15}K_5 = -0{\cdot}230755-13{\cdot}239893\,n''$$

giving (*cf.* equation (153)),

$$s = -0{\cdot}1586814-13{\cdot}239893\,n''.$$

Discussion of the Figure.

35. The boundary of the pear-shaped figure, as far as the second order of small quantities has been found to be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1+e\,x\left(\alpha\frac{x^2}{a^6}+\beta\frac{y^2}{a^2b^4}+\gamma\frac{z^2}{a^2c^4}+\frac{\kappa}{a^2}\right)$$
$$+\tfrac{1}{4}e^2\left(\frac{Lx^4}{a^8}+\frac{My^4}{b^8}+\frac{Nz^4}{c^8}+\frac{2ly^2z^2}{b^4c^4}+\frac{2mz^2x^2}{c^4a^4}+\frac{2nx^2y^2}{a^4b^4}+\frac{2px^2}{a^4}+\frac{2qy^2}{b^4}+\frac{2rz^2}{c^4}+s\right) = 0 \quad . \quad (160)$$

In this equation all the coefficients have been determined; the coefficients p, q, r and s have been found to involve n'', defined in § 25 by the equation $\frac{\omega^2}{2\pi\rho} = n+e^2n''$, while the remainder are pure numbers.

Let us put

$$p = p_0 + p_1 n''$$

and similarly for q, r, and s, and let us put $e^2 n'' = \zeta$; then the equation may be put in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1+ex\left(\alpha\frac{x^2}{a^6}+\beta\frac{y^2}{a^2b^4}+\gamma\frac{z^2}{a^2c^4}+\frac{\kappa}{a^2}\right)$$
$$+\tfrac{1}{4}e^2\left(\frac{\mathrm{L}x^4}{a^8}+\frac{\mathrm{M}y^4}{b^8}+\frac{\mathrm{N}z^4}{c^8}+\frac{2ly^2z^2}{b^4c^4}+\frac{2mz^2x^2}{c^4a^4}+\frac{2nx^2y^2}{a^4b^4}+\frac{2p_0x^2}{a^4}+\frac{2q_0y^2}{b^4}+\frac{2r_0z^2}{c^4}+s_0\right)$$
$$+\tfrac{1}{4}\zeta\left(\frac{2p_1x^2}{a^4}+\frac{2q_1y^2}{b^4}+\frac{2r_1z^2}{c^4}+s_1\right)=0 \quad (161)$$

For any values whatever of e and ζ, provided only that they are sufficiently small, equation (161) will give a figure of equilibrium. If we put $e = 0$, but retain ζ, the equation becomes

$$\frac{x^2}{a^2}\left(1+\tfrac{1}{2}\zeta\frac{p_1}{a^2}\right)+\frac{y^2}{b^2}\left(1+\tfrac{1}{2}\zeta\frac{q_1}{b^2}\right)+\frac{z^2}{c^2}\left(1+\tfrac{1}{2}\zeta\frac{r_1}{c^2}\right)=1-\tfrac{1}{4}s_1\zeta,$$

which is an ellipsoid of semi-axes a', b', c', given by

$$\frac{a'^2}{a^2}=1+\tfrac{1}{2}\zeta\left(\frac{p_1}{a^2}+\tfrac{1}{2}s_1\right),\ \frac{b'^2}{b^2}=1+\tfrac{1}{2}\zeta\left(\frac{q_1}{b^2}+\tfrac{1}{2}s_1\right),\ \frac{c'^2}{c^2}=1+\tfrac{1}{2}\zeta\left(\frac{r_1}{c^2}+\tfrac{1}{2}s_1\right),$$

or, numerically,

$$\frac{a'^2}{a^2}=1+12{\cdot}71347\zeta,\ \frac{b'^2}{b^2}=1-9{\cdot}20894\zeta,\ \frac{c'^2}{c^2}=1-3{\cdot}50453\zeta.$$

It is at once clear that as ζ varies this ellipsoid coincides with the various Jacobian ellipsoids near to the standard ellipsoid.

If we put $\zeta = 0$ but retain e in equation (161) we obtain equation (160) with $n'' = 0$; *i.e.* we obtain a series of figures of equilibrium all having the same angular velocity as the standard Jacobian ellipsoid from which they are derived.

The two series of figures obtained by putting $e = 0$ and $\zeta = 0$ in equation (161) may be represented by the two intersecting straight lines POP′, QOQ′ in fig. 1, the point of bifurcation being of course represented by the point O. The general figure of equilibrium represented in equation (161) is, however, arrived at by assigning values to both e and ζ, these values being limited by the condition only that e and ζ shall be so small that e^3 and $\zeta^{3/2}$ shall be negligible. Thus the figures of equilibrium given by equation (161) are represented by all the points inside a certain rectangle ABCD surrounding the point O. They do not fall into two linear series, as it seems to be tacitly assumed by DARWIN and POINCARÉ that they will do.

36. The circumstance that the two linear series lose their identity and become merged indistinguishably into a general area seems to be predicted as a direct consequence of

POINCARÉ's general analysis, coupled with the linearity of the equations ($\nabla^2 W = -4\pi\rho$, &c.) which lead to figures of equilibrium. For the vanishing of the Hessian ($\Delta = 0$ in the notation of POINCARÉ *) which expresses the condition that a point of bifurcation should exist, expresses also the condition that the two linear series should be merged with an *area* of linear series as they approach the point of bifurcation.†

There is, of course, no question that in the neighbourhood of the point O two linear series do actually exist, such as may be represented by the lines POP′, ROR′ in fig. 1; this is abundantly proved by POINCARÉ's general argument.‡ What is now maintained is that an expansion as far as e^2 only, does not suffice to reveal the

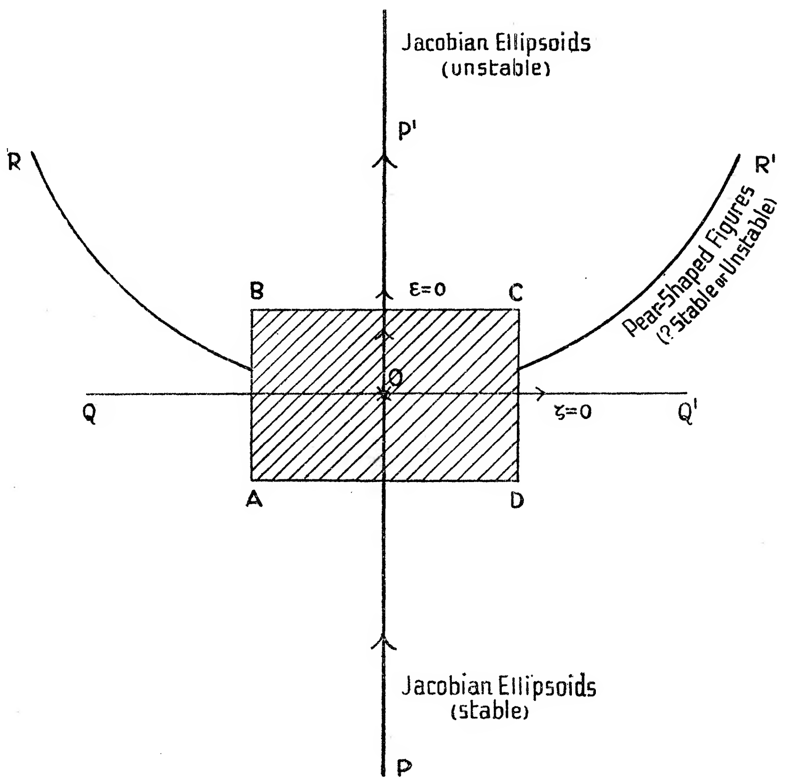

Fig. 1.

directions in which these linear series start out from the point of bifurcation. So long as our vision is limited to the interior of the rectangle ABCD in fig. 1, we can know nothing of the direction in which the line OR starts out from O. And the whole difficulty is merely one introduced by the artificial method of expansion in powers of a parameter; as soon as this artificial method is abandoned the rectangle ABCD shrinks to an infinitesimal size, and the curves POP′ and ROR′ become merely two lines intersecting in the point O without any complications. An exactly

* "Sur l'equilibre d'une masse fluide animée d'un mouvement de rotation," 'Acta Math.,' VII., p. 259.

† *Cf.* footnote to p. 74.

‡ *Loc. cit.*, § 2.

analogous situation arises in considering the directions in which lines of force start out from a point of equilibrium in an electrostatic field.*

37. A more interesting illustration of the difficulty will be found in an investigation of the figures of equilibrium of rotating liquid cylinders which I published in 1902.† In this paper the equation of the cross-section of a figure of equilibrium corresponding to a rotation ω is supposed to be expressed in the form‡

$$\xi = \left(1 - \frac{\omega^2}{2\pi\rho}\right)\left(\tfrac{8}{5}\xi_0 + \xi_1\theta + \xi_2\theta^2 + \xi_3\theta^3 + \ldots\right), \quad . \quad . \quad . \quad . \quad . \quad (162)$$

where ξ_0, ξ_1, ξ_2, ξ_3 ... are functions of η; ξ, η are complex co-ordinates given by $\xi = x+iy$, $\eta = x-iy$, and x, y are ordinary Cartesian co-ordinates measured from the axis of rotation. The quantity θ is a parameter, analogous to the e of the present paper, measuring distance from the point of bifurcation at which the pear-shaped figure and the elliptic cylinder coalesce. At this point of bifurcation, $1-\frac{\omega^2}{2\pi\rho} = \frac{5}{8}$, so that $\xi = \xi_0$, of which the value is shown to be

$$\xi_0 = \tfrac{5}{4}\eta \pm \tfrac{1}{2}\sqrt{\left(\tfrac{9}{4}\eta^2 - 10a^2\right)}.$$

* If V is the potential of an electrostatic field, the equation of a line of force will be

$$\frac{\frac{\partial V}{\partial x}}{l} = \frac{\frac{\partial V}{\partial y}}{m} = \frac{\frac{\partial V}{\partial z}}{n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.})$$

where l, m, n are direction-cosines. Two lines of force will meet in a point of equilibrium (just as two linear series meet in a point of bifurcation), and the condition for this is

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial z} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii.})$$

Let x_0, y_0, z_0 be a point of equilibrium satisfying (ii.), then, if e is a small quantity of the first order, the point

$$x_0 + \lambda e, \quad y_0 + \mu e, \quad z_0 + \nu e$$

will, as e varies from zero upwards, trace out a line passing through x_0, y_0, z_0. The condition that this shall be a line of force is, as far as first powers of e,

$$\frac{\frac{\partial V}{\partial x}}{\lambda} = \frac{\frac{\partial V}{\partial y}}{\mu} = \frac{\frac{\partial V}{\partial z}}{\nu},$$

and this is satisfied (analytically) because of equations (ii.) for all values of λ, μ, ν. Thus, as far as first powers of e, there are as many lines of force through the point of equilibrium as there are values of λ, μ, ν; an infinite number. But on going as far as e^2, it becomes clear that there are only two true lines of force through this point. The condition that a point of equilibrium shall exist is also the condition that, if the approximations are not carried far enough, there shall be the confusion of an infinite number of lines appearing to satisfy the condition for a line of force, and the analogous statement is true for points of bifurcation and linear series of figures of equilibrium.

† "On the Equilibrium of Rotating Liquid Cylinders," 'Phil. Trans.,' A, 200, p. 67.

‡ *Loc. cit.*, equation (71), p. 86.

The calculation of ξ_1 presents no difficulty, and equation (162) as far as ξ_1 only is shown to be the equation of a pear-shaped figure. On calculating ξ_2 its value is found to be of the form (*cf.* § 22 of the "Cylinders" paper),

$$\xi_2 = \frac{180}{7}\eta^3 - \frac{2825}{28}\eta - \frac{4375}{27}\eta^{-3} + \ldots + \delta_2\left(\frac{48}{25}\eta + \frac{1984}{81}\eta^{-3} + \ldots\right)$$

where δ_2 is analogous to the n'' of the present paper; to be exact the rotation for any value of θ is supposed given by

$$1 - \frac{\omega^2}{2\pi\rho} = \tfrac{5}{8} + \delta_2\theta^2 + \delta_3\theta^3 + \ldots.$$

Again, then, as far as θ^2 there is a doubly-infinite series of figures of equilibrium, not two singly-infinite series.

In this earlier and simpler investigation, it was an easy matter to carry the computations to the third, fourth, and fifth orders of small quantities. It was found that the equations giving ξ_3 for a figure of equilibrium could not be satisfied so long as δ_2 was kept indeterminate, they could only be satisfied for one special value of δ_2, namely $\delta_2 = -\frac{8625}{448}$. After having determined the value of δ_2 in this way, it was possible to investigate the stability, and the pear-shaped cylinder was found in point of fact to be stable. What is important in connection with the present paper is that it was not possible to determine the stability of the pear until after its figure had been determined to the *third order* of small quantities.*

38. The work of POINCARÉ can hardly be compared in detail with the investigation of the present paper, because POINCARÉ tacitly assumes the whole point at issue; namely, that it is possible to determine the beginning of the pear-shaped series from an investigation of figures of equilibrium going only as far as second-order terms. The work of DARWIN admits of detailed comparison, because DARWIN'S work claims actually to have effected the determination which I am compelled to believe, after my investigation, to be impossible.

It will be clear that any extra condition in addition to the conditions that the figure should be one of equilibrium will provide an additional equation which will reduce the doubly-infinite series inside the rectangle ABCD down to a singly-infinite series represented by a straight line. For instance, the condition that the angular velocity shall remain constant would reduce the rectangle to the straight line QOQ'; the condition that the angular momentum should remain constant would reduce it to

* The previous investigation on cylinders and the present one on three-dimensional bodies follow widely different methods; the present paper is in no sense a translation of the former from two into three dimensions. The two papers were written at an interval of twelve years, and I had hardly referred to the former paper in writing the present one until after I had encountered the difficulty of not being able to determine the stability from the second-order figure; I then discovered that precisely the same situation had arisen in my former investigation.

some curve at present undetermined. Hence, if the present paper is sound, we should anticipate that DARWIN obtained a linear series instead of a rectangle of configurations by the introduction of some adventitious condition not necessary to equilibrium.

39. DARWIN supposes his pear defined as far as the second order by the equation

$$\tau = c - e\mathrm{S}_3 - \sum_1^\infty f_i^s \mathrm{S}_i^s, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (163)$$

where τ gives a measure of the surface displacement, e is a parameter analogous to our e (although not numerically the same), and the quantities f_i^s are coefficients which must vary as we pass along the linear series, but are constants, as is also e, for any single figure of equilibrium. The quantities S_3, S_i^s are ellipsoidal harmonics. DARWIN supposes e to be a small quantity of the first order, and the coefficients f_i^s to be small quantities of the second order. The energy E of the distorted ellipsoid will differ from that of the original ellipsoid from which the displacement τ is measured by a small quantity $\delta\mathrm{E}$ which will involve e, f_i^s and $\delta\omega^2$, where $\delta\omega^2$ is the increase in the value of ω^2 for the distorted figure.

The first order displacement of the ellipsoid will be represented by the first terms

$$\tau = c - e\mathrm{S}_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (164)$$

of equation (163), and this is supposed to be a possible figure of equilibrium with $\delta\omega^2 = 0$. The corresponding value of $\delta\mathrm{E}$ will be of the form

$$\delta\mathrm{E} = \alpha e^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (165)$$

and the coefficient α must therefore vanish if the original ellipsoid was one corresponding to a point of bifurcation.

If we do not at present assume that $\alpha = 0$, the value of $\delta\mathrm{E}$ arising from the second-order displacement (163) will be of the form

$$\delta\mathrm{E} = \alpha e^2 + \beta e^4 + \gamma e^2 f + \theta f^2 + \frac{\delta\omega^2}{2\pi\rho}\left(a + be^2 + fc\right) \quad . \quad . \quad . \quad . \quad . \quad (166)$$

in which a single term in f has been taken as being typical of all the terms in the coefficients f_s^i. The condition that the displacement (163) together with an increase $\delta\omega^2$ in ω^2 shall give rise to a figure of equilibrium is that E shall be stationary for all variations of e and f; it is expressed by the equations

$$\frac{\partial}{\partial e}(\delta\mathrm{E}) = 0, \qquad \frac{\partial}{\partial f}(\delta\mathrm{E}) = 0, \qquad (f = f_s^i, \text{ \&c.}). \quad . \quad . \quad . \quad . \quad (167)$$

Expression (166) is the same in form as that given by DARWIN ('Coll. Works,' vol. 4, p. 349), except that he omits the term αe^2 from $\delta\mathrm{E}$; and equations (167) are

the same in form as those from which DARWIN obtains the conditions of equilibrium. The equation $\frac{\partial}{\partial e}(\delta \mathrm{E}) = 0$, written out in full, becomes

$$\alpha + 2\beta e^2 + \gamma f + b\frac{\delta\omega^2}{2\pi\rho} = 0. \qquad (168)$$

If second-order terms are neglected equation (168) reduces to

$$\alpha = 0, \qquad (169)$$

and if α is taken equal to zero, equation (168) reduces to

$$2\beta e^2 + \gamma f + b\frac{\delta\omega^2}{2\pi\rho} = 0. \qquad (170)$$

DARWIN'S method is in effect to replace equation (168) by equations (169) and (170). This, in my opinion, introduces one limitation too many on the values of the variables, and so reduces the doubly-infinite series of figures of the second order to two singly-infinite series. Equation (169) must undoubtedly be true if second-order terms are neglected, but we may take

$$\alpha = \lambda, \qquad 2\beta e^2 + \gamma f + b\frac{\delta\omega^2}{2\pi\rho} = -\lambda \qquad (171)$$

where λ is a quantity of the second order entirely at our disposal, and still satisfy all the conditions necessary for equilibrium. I think it will be found that DARWIN'S procedure in effect introduces the spurious condition $\lambda = 0$. DARWIN'S equations are of course *sufficient* to ensure equilibrium; what is maintained is that they are not *necessary* and so do not disclose all possible figures of equilibrium.

In this way I believe it will be found that DARWIN has limited himself to one linear series of figures ($\lambda = 0$) instead of the doubly-infinite series represented inside the rectangle ABCD in fig. 1. If this series should happen to run on continuously at the edge of the rectangle with the true series ROR′ in fig. 1, then of course DARWIN'S investigation would stand. But no reason suggests itself, and certainly DARWIN (not foreseeing the complication of the doubly-infinite series) gives no reason, why this should be the case. For some value of λ the two series will run on continuously, but there is no assignable reason why this value should be $\lambda = 0$.

What, then, will happen if we try to carry DARWIN'S approximation on to third-order terms by his method? I think it will be found that his series comes to a dead end before the third-order terms are arrived at, precisely as if it ran on to the boundary of the rectangle ABCD in fig. 1, and could get no further. If the displacement goes as far as third-order terms, $\delta\mathrm{E}$ must go as far as sixth-order terms, and will contain terms of orders 2, 4, and 6, say

$$\delta\mathrm{E} = \delta\mathrm{E}_2 + \delta\mathrm{E}_4 + \delta\mathrm{E}_6.$$

The equations necessary for equilibrium are

$$\frac{\partial}{\partial e}(\delta E_2 + \delta E_4 + \delta E_6) = 0, \qquad \frac{\partial}{\partial f}(\delta E_2 + \delta E_4 + \delta E_6) = 0, \qquad (f = f_s^i, \&c.), \qquad (172)$$

while the equations provided by DARWIN's methods would be

$$\frac{\partial}{\partial e}(\delta E_2) = 0, \qquad \frac{\partial}{\partial f}(\delta E_2) = 0,$$
$$\frac{\partial}{\partial e}(\delta E_4) = 0, \qquad \frac{\partial}{\partial f}(\delta E_4) = 0,$$
$$\frac{\partial}{\partial e}(\delta E_6) = 0, \qquad \frac{\partial}{\partial f}(\delta E_6) = 0,$$

and it will readily be seen that there are more equations than can be satisfied by the variables at our disposal. The conclusions of this section are put forward with the utmost diffidence, but to the present writer they seem inevitable.

40. Assuming that no glaring error has been made, the present investigation seems to indicate that so far the knowledge we have as to the stability of the pear-shaped figure amounts to absolutely nothing. The required knowledge can only be obtained by carrying the figure of the pear to a still higher approximation. In the parallel investigation on cylinders it was found that the stability could be examined as soon as the figure was determined as far as third-order terms, and doubtless the same will prove to be the case in the present problem. Fortunately the method of the present paper is one which lends itself to indefinite extension, limited only by labour of computation, so that it ought to be possible to proceed to third-order terms and determine the stability of the pear, and if the pear then proves to be stable, to proceed to higher orders and so examine the series of pear-shaped forms. In the previous investigation on cylindrical figures it was found that an expansion as far as fifth-order terms gave a good approximation to the pear-shaped figure up to a stage where it was obviously just about to divide into two separate masses. It is only in the hope that I shall be able to carry the present investigation further, that I have ventured to put forward the present somewhat lengthy piece of work which has so far led only to such negative and disappointing conclusions.